§0

rough path.

- Integral f ? $\int_0^t f(s)\,dB_s(\omega)$, $s\in[0,T]$. B_s : Brownian motion.

① Riemann. $\int_0^t f(s)ds = \lim_{|\Delta s|\to 0}\sum_i f(s_i)(s_{i+1}-s_i)$. Lebesgue. $\int_0^t f(s)ds = \sup\{\int_0^t h(s)ds,\ h \text{ simple},\ h\le f\}$ $(f\geq 0)$

② Riemann–Stieltjes. $\cdots \sum_i f(s_i)(g(s_{i+1})-g(s_i))$. Lebesgue–Stieltjes. $g\mapsto \mu_g$ (signed)

$g\in BV$. ($\Leftrightarrow$ $\Sigma\ \exists$, for $\forall f$ cont.)

by chording thm.

③ $(B_t)_{t\geq 0}$ $t\longmapsto B_t(\omega)$ non BV.

by Itô.

RS $\sum_i f(t_i)(B_{t_{i+1}}(\omega)-B_{t_i}(\omega)) \xrightarrow[|\Delta t|\to 0]{P.} \int_0^t f(s)\,dB_s(\omega)$

- Preparation

Monotone Class. Dynkin-sys. 3 conv Thms $\left|\frac{\partial f(t,x)}{\partial t}\right|\leq g(x)\in L^1$.

Prob. meas. completize. Carathéodory expansion. $\mu\ll\nu$ σ-fin. Fubini. $f(x,y)\geq 0$. $f\in L^1$.

Regularity (outer & inner) (sep.) several conv. L^p-. charateristic.$\mathcal{F}$. gen. $\mathcal{L}$.

wk.~ – $\mathcal{F}$ - p.w./ inc. uni.

$\xi\in\mathbb{N}$, $G_\xi(z) = \sum_{n=0}^\infty P(\xi=n)z^n$, $|z|\leq 1$, $z\in\mathbb{C}$.

$G(1)=1 \underset{\text{Abel}\ \xi}{\longrightarrow} G'(z) = \sum_{n=1}^\infty n z^{n-1} P(\xi=n)$, $|z|<1$. uniquely determined. by $P(\xi=n) = \frac{G_\xi^{(n)}(z)}{n!}$. moment if $\mathbb{E}\xi<\infty$, $\mathbb{E}\xi = \lim_{z\to 1} G'_\xi(z)$.

$\mathcal{L}\mu(t) = \int_0^\infty e^{-tx}\mu(dx)$, $\forall t\geq 0$ $\Downarrow$ (Pf) $\mu(x)$.

(for $\mathbb{N}\ni\xi\overset{d}{=}\mu$, $\mathcal{L}\mu(t) = \mathbb{E}e^{-t\xi}$ let $z=e^{-t}\in(0,1]$ then $=G_\xi(z)$.)

- <u>Lem</u>

uni. intgl. $\Leftrightarrow$ L^1-bounded + uni. abs. cont. (Δδ uniformity) (equi-)

($\lim_{n\to\infty}\sup_{i\in I}\mathbb{E}(|\xi_i|;|\xi_i|\geq n)=0$) ($\sup_i\mathbb{E}|\xi_i|<\infty$) ($\forall\varepsilon>0$, $\exists\delta>0$, s.t. $\forall i\in I$, $\forall P(A)<\delta$, $\Rightarrow \mathbb{E}(|\xi_i|;A)<\varepsilon$).

Pf: ($\Rightarrow$) $\mathbb{E}(|\xi_i|;A) = \mathbb{E}(|\xi_i|;A\cap\{|\xi_i|\geq N\}) + \mathbb{E}(|\xi_i|;A\cap\{|\xi_i|\leq N\}) \leq \underbrace{\mathbb{E}(|\xi_i|;|\xi_i|\geq N)}_{<\frac{\varepsilon}{2}} + N\underbrace{P(A)}_{\frac{\varepsilon}{2N}.}$ $<\varepsilon$.

($\Leftarrow$) $\forall\varepsilon>0$, $\exists N_0>0$, s.t. $N\geq N_0$, $\mathbb{E}(|\xi_i|, |\xi_i|\geq N)<\varepsilon$, $\forall i\in I$.

by Cheb. $P(|\xi_i|\geq N)\leq\frac{1}{N}\mathbb{E}|\xi_i|$. See $A=\{|\xi_i|\geq N\}$ in abs. cont.

<u>Thm</u> $\xi_n\xrightarrow{L^1}\xi \Leftrightarrow \xi_n\xrightarrow{P}\xi$, $\{\xi_n\}$ uni. intgl.

Pf: ($\Rightarrow$) ① $\sup_i\mathbb{E}|\xi_i|<\infty$ since $\|\xi_n-\xi\|\to 0 \overset{\text{triangle}}{\Longrightarrow} \|\xi_n\|\to\|\xi\|$.

P. is obv. ② $\forall\varepsilon>0$, $\exists\delta>0$, s.t. $\cdots$, $\mathbb{E}(|\xi_i|;A)\leq\mathbb{E}(|\xi_i-\xi|,A)+\mathbb{E}(|\xi|,A)<\varepsilon$ $P(A)<\delta$ ($\exists N$, $\forall i\geq N$, $<\frac{\varepsilon}{2}$ (choose enough δ_0) and for $i\leq N-1$, choose enough δ_{N-1}) s.t. $<\frac{\varepsilon}{2}$. $\delta=\delta_0\wedge\delta_1\cdots\wedge\delta_{N-1}$.

($\Leftarrow$) $\forall\varepsilon>0$, $\mathbb{E}|\xi_n-\xi| =$

$\mathbb{E}(|\xi_n-\xi|, |\xi_n-\xi|>\varepsilon)+\mathbb{E}(\cdots<\varepsilon) \leq \mathbb{E}(|\xi_n|;>\varepsilon) + \mathbb{E}(|\xi|;>\varepsilon) + \varepsilon$

$\underset{2}{<}\varepsilon$ u. abs. cont. p. conv.

for L^p ✓. $\xi_n\xrightarrow{L^p}\xi \Leftrightarrow \xrightarrow{P.} + \{|\xi_n|^p\}$ u. intgl.

$(0<p<\infty)$

Thm. $\xi_n \xrightarrow{P.} \xi$ equiv.: (Vitali conv.) (Vitali $\Rightarrow$ DCT.)

1. $\xi_n \xrightarrow{L^p} \xi$

2. $\{|\xi_n|^p\}$ uni. intgl.

3. $\lim_{n\to\infty} \mathbb{E}|\xi_n|^p = \mathbb{E}|\xi|^p$.

(esp. for $p=1$.)

(1. $\to$ 3. is easy for $p\geq 1$)

(Pf 3. $\to$ 1. 2.)

$f_n \xrightarrow{a.e.} f \in L^1 \underset{u.\ intgl.}{\overset{|f_n|\leq g\in L^1}{\not\Rightarrow}}$ e.i. f_n.

• abs. cont. (of intg.) $f \in L^1$.

$f_n = f\wedge n$, $f_n \xrightarrow{P.w.} f$. DCT $\int |f-f_n|\,dm \leq \frac{\varepsilon}{2}$

(ceiling) $\int_A |f|\,dm \leq \int_A |f-f_n|\,dm + \int_A |f_n|\,dm$

(or by $|f-s|+|s|$ $\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2n}\cdot n$

$s = \sum_1^N g_i \chi_{A_i}$, $G = \max_i g_i$.) $\quad = \varepsilon$.

uni. intgl. $\Leftrightarrow$ $\{P_{\xi_i}\}_{i\in I}$ tight.

§1

conditional exp.

uniqueness: $\mathbb{E}(\eta_1-\eta_2; A) = 0, \forall A \in \mathcal{A}$. $\eta_1 - \eta_2 \in \mathcal{A}$, $A = \{\eta_1 \neq \eta_2\}$.

existence: $\mu(A) = \mathbb{E}(\xi; A)$ is a well-defined signed meas.

$\mu \ll \mathbb{P}_A \xrightarrow{R-N} \exists \eta \in L^1(\Omega, \mathcal{A}, \mathbb{P}_A)$ s.t. $\mu = \eta\cdot\mathbb{P}_A$.

(or geometry in L^2 then lim)

• $\mathbb{E}\xi = \mathbb{E}(\mathbb{E}(\xi|\mathcal{A}))$

($= \eta\cdot\mathbb{P}_{\mathcal{A}}(\Omega) = \mu(\Omega) = \mathbb{E}\xi$)

$\exists \eta \in L^1$, $|\xi_n| \leq \eta$, $\forall n$. $\xi_n \xrightarrow{a.s.} \xi$,

(DCT) $\lim_{n\to\infty} \mathbb{E}(\xi_n|\mathcal{A}) = \mathbb{E}(\lim_{n\to\infty}\xi_n|\mathcal{A})$.

Pf: ξ intgl. by Fatou $\leq \mathbb{E}\eta$.

$\sup_{n\geq k} |\mathbb{E}(\xi_n|\mathcal{A}) - \mathbb{E}(\xi|\mathcal{A})| \leq \sup_{n\geq k} \mathbb{E}(|\xi_n - \xi||\mathcal{A})$

$\leq \mathbb{E}(\sup_{n\geq k}|\xi_n-\xi||\mathcal{A}) \xrightarrow{a.s.} \xi$, $\xrightarrow{L^1} 0$ thus $\varlimsup |\mathbb{E}_n - \mathbb{E}| = 0$

MCT $0 \leq \xi_n \uparrow \xi \in L^1$, $\mathbb{E}(\xi|\mathcal{A}) = \lim_n \mathbb{E}(\xi_n|\mathcal{A})$;

Fatou $\varliminf_n \xi_n \in L^1$, $\mathbb{E}(\varliminf \xi_n|\mathcal{A}) \leq \varliminf_n \mathbb{E}(\xi_n|\mathcal{A})$;

Jensen, Tower.. (Pf:

• essential bounded ($\mathbb{E}(\cdots) = \mathbb{E}(\sup_{n\geq k}|\cdot\cdot|)$

tech: $G_n = \{\xi_n \leq n\}$, $G_n \uparrow \Omega$. $\to 0$)

• Vitali $\Rightarrow$ DConv. Note for cond. exp.: $\xi_n \underset{L^1}{\overset{a.s.}{\to}} \xi \overset{?}{\Rightarrow} \lim_n \mathbb{E}(\xi_n|\mathcal{A}) = \mathbb{E}(\xi|\mathcal{A})$ ×. $\in \mathcal{F}_n$ then use DCT etc.

• Fact: ξ, η_i. $\xi \in \sigma(\eta_i) \Rightarrow \exists$ Borel f s.t. $\xi = f(\eta_i)$.

Martingale $\mathbb{E}(X_{n+m} - X_n|\mathcal{F}_n) = 0$ (by tower). ϕ convex. then $\phi(X_n)$ (if intgl.) submg.

(for X_n submg. ϕ need $\nearrow$)

$H_n \in \mathcal{F}_{n-1}$, $Y_n - Y_{n-1} = H_n(X_n - X_{n-1})$, $n\geq 1$

Doob's Thm. Y_n intgl. $\Rightarrow$ Y_n mg.

Stop $\min\phi \triangleq +\infty$. $T: (\Omega, \mathcal{F}) \to [0, +\infty]$. $\{T\leq n\} \in \mathcal{F}_n$.

(Not: e.g. $T = \min\{1\leq n\leq N: X_n = \max_{1\leq i\leq N} X_i\}$)

τ is stop. X_n is mg. $\Rightarrow X^\tau \triangleq X_{n\wedge\tau}$ is mg. ($X^\tau_{n+1} - X^\tau_n = 1_{\{T\geq n+1\}}(X_{n+1}-X_n)$,

Thm. $\forall$ bounded $\tau \geq \sigma$, X_n is mg. $\mathbb{E}X_\sigma = \mathbb{E}X_\tau$ $(= \mathbb{E}X_0)$. (Same for submg.) $H_{n+1} = 1_{\{T\geq n+1\}} = 1 - 1_{\{T\leq n\}} \in \mathcal{F}_n$)

($|T(\omega)| \leq N$, $\forall \omega\in\Omega$)

$(X_\tau(\omega) \triangleq X_{T(\omega)}(\omega)$ $(\because 1_{\{T<\infty\}})$, Pf: $(\Rightarrow)$ X^τ mg., $\mathbb{E}X^\tau_n = \mathbb{E}X^\tau_0 = \mathbb{E}X_0$, $\forall n\geq 0$, i.e. $\mathbb{E}X_\tau = \mathbb{E}X_0$.

$|X_\tau(\omega)| \leq \sum_{n\geq 0}^N |X_n(\omega) 1_{\{T=n\}}| \leq \sum_{n\geq 0}^N |X_n(\omega)|$ intgl.) $(\Leftarrow)$ if $\forall B \in \mathcal{F}_n$, $\mathbb{E}(X_{n+1}1_B) = \mathbb{E}(X_n 1_B)$.

let $\tau = (n+1)1_B + N1_{B^c}$, $\quad \overset{\shortparallel}{\mathbb{E}X_\tau} \quad \overset{\shortparallel}{\mathbb{E}X_\sigma}$

(submg. $Y_n = X^\tau_n - X^\sigma_n$ submg. stops $\sigma = n1_B + N1_{B^c}$,

$\Rightarrow$ since $Y_{n+1} - Y_n \overset{\sigma\leq\tau}{=} 1_{\{T\geq n+1\}\wedge\{\sigma\leq n\}}(X_{n+1}-X_n)$) $N > n+1$. (submg. $\geq$) $\Leftarrow$

Thm. (Doob Max) X nonneg. submg. (e.g. $|X|$)

$\eta \triangleq \max_{1\le n\le N} X_n$ (not stop, not controlled $\mathbb{E}\eta \le \mathbb{E}X_N X$)

① $\mathbb{P}(\eta \ge t) \le \frac{1}{t}\mathbb{E}X_N$ by Fubini . Hölder.

② $\mathbb{E}\eta^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}X_N^p$

- $X \in L^1$, flow $\mathcal{F}$. $\Leftrightarrow$ $X_n = \mathbb{E}(X|\mathcal{F}_n)$ is a UI mg. ($\Leftarrow$ by Doob upcrossing ineq., X_n mg. + L^1-bd. $\to X_n \xrightarrow{a.s.} X_\infty$)

UI : **Thm.** equiv.:

1. $\{X_n\}$ UI mg.
2. $\exists X_\infty \in L^1$ s.t. $X_n \xrightarrow[L^1]{a.s.} X_\infty$. (Vitali. p. from a.s. from mg.) $1.\to 2.$:
3. $\exists X_\infty \in L^1$ s.t. $X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)$.

$2.\to 3.$ $\left(|\mathbb{E}_n - \mathbb{E}_\infty| \le \mathbb{E}(|_n - _\infty| \,|\mathcal{F}_n) \le \mathbb{E}|_n - _\infty| \to 0\right)$

($3.\to 1.$ UI conditions) subseq. not Vitali!

§2

- right-conv.:

$\sup_n \mathbb{E}|X_n| < \infty$ makes $X_\infty \in L^1$; or $\sup_n \mathbb{E}X_n^+ < \infty$ only gives $X_n \xrightarrow{a.s.} X_\infty$.

- guidline of discrete mg.:

1. intg. $(H.X)_n - (H.X)_{n-1} = H_n(X_n - X_{n-1})$; stop. X^τ.

2. desc. of mg.:

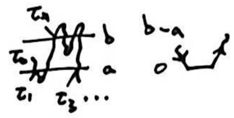
mg. $\Leftrightarrow$ bd τ, $\mathbb{E}X_\tau = \mathbb{E}X_0$.

submg. $\Leftrightarrow$ bd. $\tau \le \sigma$, $\mathbb{E}X_\tau \le \mathbb{E}X_\sigma$.

$\Leftrightarrow$ bd. $\tau \le \sigma$, $\mathbb{E}X_\tau \le \mathbb{E}(X_\sigma|\mathcal{F}_\tau)$.

(similar pf to above)

3. two ineqs.



- $\eta = \max_{n\le N} X_n$. $\mathbb{P}(\eta \ge \lambda) \le \frac{1}{\lambda}\mathbb{E}X_N$.

$\mathbb{E}\eta^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}X_N^p$ (Fubini & Hölder.)

- upcrossing. $\sup_n \mathbb{E}X_n^+ < \infty \Rightarrow \lim_n X_n$ a.s.

$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \le n\} \in \mathcal{F}_n\}$, $X_\tau \in \mathcal{F}_\tau$.

- sq.-intgl. mg.:

$\sup_n \mathbb{E}X_n^2 < \infty$. ($p>1$ other is same)

$|X_n| \le \eta = \lim_N \eta_N \overset{MCT}{\Rightarrow} \mathbb{E}\eta^2 \le 4\sup \mathbb{E}X_n^2 < \infty$.

$\mathbb{E}\eta_N^2 \le 4\mathbb{E}X_N^2$. thus UI mg.

(right closed = UI = Doob) $\subset$ UI.

(Doob mg.) $X_n \xrightarrow[L^1]{a.s.} X_\infty$, $|X_\infty| \le \eta \in L^2$.

$\lim_n \mathbb{E}|X_n - X_\infty|^2 = 0$ then $X_n \xrightarrow[L^1, L^2]{a.s.} X_\infty$.

- reverse mg.:

$\cdots \subseteq \mathcal{F}_{-n} \cdots \subseteq \mathcal{F}_{-1} \subseteq \mathcal{F}_0 \subseteq \mathcal{F}$.

$X_{-\infty} \longleftarrow X_{-n} \cdots X_{-1}, X_0$ ($\in L^1$).

$\mathcal{F}_{-\infty} \triangleq \bigcap_n \mathcal{F}_{-n}$.

<u>UI</u> by $X_{-n} = \mathbb{E}(X_0|\mathcal{F}_{-n})$ ✓.

<u>a.s.</u> $\mathbb{E}U_{-N}^X[a,b] \le \frac{1}{b-a}\mathbb{E}(X_0 - a)^+$,

MCT. $\mathbb{E}U_{-\infty}^X[a,b] < \infty$ ✓.

(for submg. $\le$ ✓. only to check UI?)

$\underline{X_{-\infty} = \mathbb{E}(X_n|\mathcal{F}_{-\infty})}$ i.e. $X_{-n} \xrightarrow[L_1]{a.s.} X_{-\infty}$, $X_{-\infty} \overset{a.s.}{=} \lim_{N\ni n\to\infty} X_{-n} \in \bigcap_N \mathcal{F}_{-N} = \mathcal{F}_{-\infty}$. (by completeness $\mathcal{N}$)

$\forall B \in \mathcal{F}_{-\infty}$, $\mathbb{E}(X_0 1_B) = \mathbb{E}(X_{-n}1_B) \xrightarrow{\text{by } L^1} \mathbb{E}(X_{-\infty}1_B)$ ✓.

Fact: $X_n := -nX$.

+ condition equiv. $\left\{\begin{array}{l} \inf_n \mathbb{E}X_{-n} > -\infty. \\ \{X_n\} \text{ UI mg.} \\ \sup_n \mathbb{E}|X_n| < \infty. \end{array}\right.$ ✓ Pf :

$\mathbb{E}(|X_{-n}|; |X_{-n}| > N) = \mathbb{E}(X_{-n}; X_{-n} > N) + \mathbb{E}(-X_{-n}; X_{-n} < -N)$

$= \mathbb{E}(X_{-n}; X_{-n} > N) + \mathbb{E}(X_{-n}; X_{-n} \ge -N) - \mathbb{E}X_{-n}$

$\inf_n \mathbb{E}X_{-n} \ge \inf(-\mathbb{E}|X_{-n}|)$

$= -\sup \mathbb{E}|X_n| > -\infty$.

$\exists k$ s.t. $\mathbb{E}X_{-k} \le \inf_n \mathbb{E}X_{-n} + \varepsilon$ (conv. of $\mathbb{E}X_{-n}$)

then $\le \mathbb{E}(X_{-k}; X_{-n} > N) + \mathbb{E}(X_{-k}; X_{-n} \ge -N) - \mathbb{E}X_{-k} + \varepsilon$

let $-n < -k$, $= \mathbb{E}(X_{-k}; |X_{-n}| > N) + \varepsilon$ (uni. bd. $(2\mathbb{E}X_0^+ - \inf \mathbb{E}X)$)

and by $\mathbb{P}(|X_{-n}| > N) \le \frac{1}{N}\mathbb{E}|X_{-n}|$

i.e. $\mathbb{E}(X_{-k}|\mathcal{F}_{-n}) \ge X_{-n}$. $= \frac{1}{N}\mathbb{E}(2X_{-n}^+ - X_{-n}) \le \frac{\cdot}{N}$

§3

(Pf of Kol.: ① Polish E ← Radon μ. $H_n \setminus K_n$. μ on $\mathcal{E}_0^T$ is a premeas. i.e. pf cont. on $A_n \downarrow \phi$, $\mathbb{P}(A_n) \downarrow 0$.
② $C_F(\Omega) \to C(\Omega)$ SW & Rieze.) $\sigma(\mathcal{E}_0^T)$

① $\xi : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$. ② $\xi_1, \cdots \xi_n, \cdots$ i.i.d. ③ $X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to E$.

$\underline{F}$ dis. $\mathbb{R} \xrightarrow{F} ((0,1), |\cdot|)$ $\mu \times \mu \times \cdots$ $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E^T, \mathcal{E}^T, \mathbb{P} \circ X^{-1})$

$\xi = F^{-1}$.

Kolmogorov. $\mu \Leftrightarrow (\mu_I)_{I \subseteq T}$. $\phi_I \downarrow$ (canonical sp.) $(E^I, \mathcal{E}^I, \mu_I)$

Gauss process. $X_I \sim N(a, A)$. (A sys. ≥ 0.) $(e^{ia'x - \frac{1}{2} x A x'})$ $\mu_I = \mu \circ X_I^{-1}$. $(\mathcal{E}^T = \sigma(Z_t, t \in T)$, proj. $Z_t(\omega) = \omega(t)$.)

(General ~) Facts: · $X \sim GG$. $S_{n \times m} \Rightarrow X \cdot S \in \mathbb{R}^m \sim GG$. · $X^{(k)} \xrightarrow{d} X \sim GG$.

- (§2) Disc.

1) intg. $Y_n - Y_{n-1} = H_n(X_n - X_{n-1})$

2) τ stop $\Rightarrow X^\tau$ is (sub)mg.

3) adapted intgl. X is mg. $\Leftrightarrow \forall$ bd. τ, $\mathbb{E} X_\tau = \mathbb{E} X_0$. (and $X_\sigma \leq \mathbb{E}(X_\tau | \mathcal{F}_\sigma)$.)
submg. $\Leftrightarrow \forall$ bd. $\sigma \leq \tau$, $\mathbb{E} X_\sigma \leq \mathbb{E} X_\tau$.

4) Doob Max Ineq. $(\lambda \mathbb{P}(\underbrace{\max_{n \leq N} X_n}_{:= \eta} \geq \lambda) \leq \mathbb{E} X_N)$ $\mathbb{E}\eta^p \leq (\frac{p}{p-1})^p \mathbb{E} X_N^p$ (nonneg.)

5) Upcrossing ineq.

- Azuma Ineq.
($\Rightarrow Y_k = \mathbb{E}(S_N | X_1, \cdots, X_k)$, $S_k = X_1 + \cdots + X_k$) to get Hoeffding.
mg. ($\Rightarrow$ McDiarmid's)
$\mathbb{P}(|X_N - X_0| \geq \varepsilon) \leq 2e^{-\frac{2\varepsilon^2}{\sum_k c_k^2}}$. if $|X_k - X_{k-1}| \leq c_k$ a.s.
?

Cont.

1) ???

2) X^τ ? △

3) ($\Leftarrow$) $\forall B \in \mathcal{F}_s$, $\mathbb{E}(X_t 1_B) \geq \mathbb{E}(X_s 1_B)$.
$t > s$ $\tau = t 1_B + N 1_{B^c}$,
$\sigma = s 1_B + N 1_{B^c}$ $(N > t)$.
($\Rightarrow$) ? △

4) seq. $D_n \cap [0, T]$. (r. cont. to $\sup X_t$)
MCT $\mathbb{P}(\hat{\ell}_n \underbrace{\max_{t^{(n)}} X_{t^{(n)}}}_{:= \eta_n} \geq \lambda) \leq \frac{\mathbb{E} X_T}{\lambda}$.

(5) see above.)

(η_n measl. $\uparrow \eta$ measl.)
(actually $\bigcup_n \{\eta_n \geq \lambda\} \neq \{\eta \geq \lambda\}$
thus need for $> \lambda - \varepsilon$ is ✓
then by $\{\eta \geq \lambda\} = \bigcap_m \{\eta > \lambda - \frac{1}{m}\}$)

△3) Pf the 'and': $\sigma_n \in D_n$. $\sigma \leftarrow \sigma_n$. disc. Doob stop thm. gives X_{σ_n} intgl.
$\mathbb{E} X_{\sigma_n} \geq \mathbb{E} X_0$ then and X_{σ_n} is a reverse submg. $X_{\sigma_n} \xrightarrow[L_1]{a.s.} X_\sigma$ (r. cont.), X_σ intgl. (X_τ.)
(from 3) that $X^\tau := X_{\tau \wedge t}$ is (sub)mg. of $(\mathcal{F}_{\tau \wedge t})$.) ~ gives $\mathbb{E}(\tau_n ; A) \geq \mathbb{E}(\sigma_n ; A)$, $\forall n$
L_1 conv. ↓ for $A \in \bigcap_n \mathcal{F}_{\sigma_n} = \mathcal{F}_\sigma$.
$\mathbb{E}(\tau ; A) \geq \mathbb{E}(\sigma ; A)$.

△2) but stronger Pf X^τ is (sub)mg of $(\mathcal{F}_t)$.
$(t > s)$ $\mathbb{E}(X_{\tau \wedge t} | \mathcal{F}_s) \geq X_{s \wedge \tau}$ i.e. to pf $\mathbb{E}(X_{\tau \wedge t} | \mathcal{F}_s) \in \mathcal{F}_{\tau \wedge t}$.
↑?
$\mathbb{E}(X_{\tau \wedge t} | \mathcal{F}_{\tau \wedge s}) \geq X_{s \wedge \tau}$ $= \mathbb{E}(X_{\tau \wedge t} 1_{\{\tau \wedge t \geq s\}} + X_{\tau \wedge t} 1_{\{\tau \wedge t < s\}} | \mathcal{F}_s)$
$\mathcal{F}_{\tau \wedge s} = \mathcal{F}_\tau \cap \mathcal{F}_s$ (note that $(\cap \mathcal{F}_t)$ $X_{\tau \wedge t} \in \mathcal{F}_{\tau \wedge t}$)
$= X_{\tau \wedge t} 1_{\{\tau \wedge t < s\}} + 1_{\{\tau \wedge t < s\}} \mathbb{E}(X_{\tau \wedge t} | \mathcal{F}_s)$ i.e. $\in \mathcal{F}_s$.
$\in \mathcal{F}_{\tau \wedge t}$ ✓ $(1_{\{\tau \wedge t \leq u\}} \cdot 1_{\{\tau \wedge t < s\}} \mathbb{E}(X_{\tau \wedge t} | \mathcal{F}_s) \in \mathcal{F}_u)$ ✓.

- Lemmas:

· Föllmer's. dense D. $X^D_{t+} \triangleq \lim_{s \in D, s \downarrow t} X_s$.
(mod. by $X_t \to X^D_{t+}$, $\mathcal{F}_t \to \mathcal{F}_{t+}$)
(X_t) (sub)mg. (X^D_{t+}) is r. cont. (sub)mg. wrt. $(\mathcal{F}_{t+})$.
and $\mathbb{E}(X^D_{t+} | \mathcal{F}_t) \geq X_t$ equals when $t \mapsto \mathbb{E} X_t$ r. cont.

· Doob decomp. submg $X_n = Y_n + Z_n$ (mg. predt and $0 \leq Z_1 \leq \cdots$.)

· $\mathcal{F}_{\tau \wedge \sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma$, $\{\tau < \sigma\}, \{\tau \leq \sigma\} \in \mathcal{F}_{\tau \wedge \sigma}$.

· $\tau \in \mathcal{F}_\tau$. r.v. $\xi \in \mathcal{F}_\tau$ iff. $\forall t$, $\xi 1_{\{\tau \leq t\}} \in \mathcal{F}_t$.

· $\tau_n \uparrow \tau$ is stop. $\tau_n \downarrow \tau$ and $\mathcal{F}_t$ r. cont. is stop.
$\sigma \leq \tau$, $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. (and then $\bigcap_n \mathcal{F}_{\tau_n} = \mathcal{F}_\tau$)

· Depiction of
i) $\lim_{t \to +\infty} X_t$ $\exists$ by L_1.
ii) $\exists X_\infty \in L^1$ s.t. $X_t = \mathbb{E}(X_\infty | \mathcal{F}_t)$
iii) $\{X_t\}_{t \geq 0}$ UI.

at this time, $X_t \xrightarrow[L^1]{a.s.} X_\infty$. esp., if $\sup_{t \geq 0} \mathbb{E}|X_t|^p < \infty$ then it satisfies and $\xrightarrow{L^p}$.

Pf: (esp.) by UI i.e. $\sup_t \mathbb{E}|X_t| < \infty$, upcrossing gives $\xrightarrow{a.s.}$.
$\sup \| \cdot \|^p < \infty$ gives UI (Hölder) and max ineq. gives
$\eta := \sup_t |X_t|^p \in L^1$ i.e. $\{|X_t|^p\}$ UI + $\xrightarrow{a.s.} \Rightarrow \xrightarrow{L^p}$.
(former 3) iii → i : Vitali ✓. (firstly UI gives $\xrightarrow{a.s.}$)
ii → iii : obvious. ✓. i → ii : $X_t \xrightarrow{L^1} X_\infty$ (not Vitali)
$\mathbb{E}|\eta_n - \eta| \leq \mathbb{E}(\mathbb{E}(|X_n - X_\infty|\,|\mathcal{F}_t)) = \mathbb{E}|X_n - X| \to 0$. by Jessen $\eta_{n_k} \xrightarrow{a.s.} \eta$ (mg. then to whole seq.) $X_t = \mathbb{E}(X_{t+s} | \mathcal{F}_t) \xrightarrow{L^1} \mathbb{E}(X_\infty | \mathcal{F}_t)$.
$\eta_n := \mathbb{E}(X_n | \mathcal{F}_t)$
$\eta := \mathbb{E}(X_\infty | \mathcal{F}_t)$.

• Cont. mg. $T=[0,+\infty)$ $(\Omega,\mathcal{F},\mathbb{P})$ complete. $\sigma(X_s; s\le t)\subseteq\mathcal{F}_t$. $\mathcal{F}_s\subseteq\mathcal{F}_t\subseteq\mathcal{F}$, $\forall s\le t$. $Y\in\mathcal{F}_\tau \iff Y\cdot 1_{\{\tau\le t\}}\in\mathcal{F}_t$.

$\tau:(\Omega,\mathcal{F})\to T\cup\{\infty\}$. $\{\tau\le t\}\in\mathcal{F}_t$, $\forall t\in T$. $\mathcal{F}_\tau=\{A\in\mathcal{F}: A\cap\{\tau\le t\}\in\mathcal{F}_t, \forall t\}\subseteq\mathcal{F}$. $X_\tau\in\mathcal{F}_\tau$.

right cont. flow $\mathcal{F}_{t+}\triangleq\bigcap_{s>t}\mathcal{F}_s=\mathcal{F}_t$. right/left cont. process $t\mapsto X_t(\omega)$. (orbit) $(\{X_\tau\in B\}=(\bigcup_k\{X_k\in B,\tau=k\})\cap\{\tau\le t\}\in\mathcal{F}_t.)$

① example of stops?

$\tau_a=\inf\{t\ge 0: |X_t|>a\}$?

(boundrize)

...

② (X_t). $X_t(\omega)$ measl.?

$X_\tau(\omega)=X_{\tau(\omega)}(\omega)$ measl.?

$X_\tau:\Omega_\tau=\{\omega\in\Omega: \tau(\omega)<\infty\}\to\mathbb{R}$, $\mathcal{F}\cap\Omega_\tau$ measl.?

③ $\mathbb{E}\int_0^T f(X_t)dt=\int_\Omega\int_0^T f(X_t,\omega)\,dt\,\mathbb{P}(d\omega)$

$F(t,\omega)$ measl.?

• Adapted Intgl. Submg. X

$\iff\forall$ bd. $\sigma\le\tau$, $\mathbb{E}X_\sigma\le\mathbb{E}X_\tau$.

$\iff\forall$ bd. $\sigma\le\tau$, $X_\sigma\le\mathbb{E}(X_\tau|\mathcal{F}_\sigma)$.

(Pf of 2.→3.: $\tilde\tau=\tau 1_A+N1_{A^c}\ge\tilde\sigma=\sigma 1_A+N1_{A^c}$, $\forall A\in\mathcal{F}_\sigma$, $N>\sigma\vee\tau$.)

Assumption: i. normal condition $\{$ $N\subseteq\mathcal{F}$ null $\Rightarrow N\subseteq\mathcal{F}_0(\subseteq\mathcal{F}_t)$. ($\mathcal{P}$ prog. set also can def.) ; right cont. $\mathcal{F}_{t+}=\mathcal{F}_t$.

∘Adapted ⊃ measl.

ii. $X:([0,T]\times\Omega,\ \mathcal{B}([0,T])\times\mathcal{F}_T)\to E$. $\forall T>0$. Leb×$\mathbb{P}$. ⊃ prog. measl. ⊃ r. cont. adapted

$(t,\omega)\mapsto X_t(\omega)$. measl. (actually $\mathcal{B}[0,T]\times\mathcal{F}_T$-measl.) stair $\lim_n X_s^{(n)}\xrightarrow{a.s.}X$.

Effectiency: Thm. $A\in\mathcal{B}(E)$, $D_A=\inf\{t\ge 0: X_t\in A\}$ is $(\mathcal{F}_t)$-stop.

$T_A=\inf\{t>0: X_t\in A\}$

$=\lim_n\inf\{t\ge\frac{1}{n}: X_t\in A\}$

(if A open, X right cont.,

$D_A=T_A$.

$\{T_A<t\}\in\mathcal{F}_t$, $\{T_A\le t\}\overset{?}{\in}\mathcal{F}_t$.

(flow cont.) Ass.ii.

if A closed, X cont.,

$\{D_A\le t\}\overset{a.s.}{=}\{\inf_{r\in[0,t]\cap\mathbb{Q}} d(X_r,A)=0\}$

Ass.i.

Thm. $X_\tau:(\Omega_\tau,\mathcal{F}_\tau\cap\Omega_\tau)\to E$. measl. $\Omega_\tau\triangleq\{\tau<+\infty\}$.

to pf $\forall B\in\mathcal{B}(E)$, $\{X_\tau\in B\}\cap\Omega_\tau\in\mathcal{F}_\tau\cap\Omega_\tau$

$\iff\forall t$, $\{X_\tau\in B, \tau<\infty\}\cap\{\tau\le t\}\in\mathcal{F}_t$.

$\iff\{X_{\tau\wedge t}\in B, \tau\le t\}\in\mathcal{F}_t$.

thus i.e. to pf $\{X_{\tau\wedge t}\in B\}\in\mathcal{F}_t$.

(note that if not cont. even $\sup_{t\le T}X_t$ may not be measl. since the σ-alg. only permits countable H_i)

$X_{\tau\wedge t}$:

$\omega\in(\Omega,\mathcal{F}_t)\xrightarrow{S(\omega)}(\tau(\omega)\wedge t,\omega)$ $X^{-1}(B)$

$(\tau(\omega)\wedge t,\omega)\in([0,T]\times\Omega,\ \mathcal{B}(E)\times\mathcal{F}_\tau)$

measl. $X(s,\omega)\triangleq X_s(\omega)$

$X_{\tau\wedge t}\in B$

• modification: submg. $t\mapsto\mathbb{E}X_t$ r. cont.

$\Rightarrow\exists(Y_t)$ r. cont. mod. $(Y_t\overset{a.s.}{=}X_t,\forall t)$ (even for a.s. r. cont. it's also true.)

strengthen: natural $\mathcal{F}_t^0=\sigma(X_s, s\le t)$. , null sets $\mathcal{N}\subset\mathcal{F}$.

let $\mathcal{F}_t=\sigma(\mathcal{F}_t^0\cup\mathcal{N})$. (much greater than completize $\mathcal{F}_t^0$)

and for $\xi\in L^1$, $\mathbb{E}(\xi|\mathcal{F}_t)\overset{a.s.}{=}\mathbb{E}(\xi|\mathcal{F}_t^0)$.

e.g. right Markov process

i) obit r. cont.

ii) sg. Markov.

Satisfies Assumptions.

∘ discrete seq. (2^n-portion)

mg. X_t $\mathcal{F}_t^0\ \cdots\ \mathcal{F}_t\ \ \mathcal{F}_{t+}$ ✓. $\overset{?}{=}$

(X_t) is $(\mathcal{F}_{t+})$-submg. if (X_t) is r. cont. $(\mathcal{F}_t)$-submg.

Pf: $k\to t$, $t_0=t$; s, t

$X_{t_n}\xrightarrow[L^1]{a.s.}X_\infty\overset{a.s.}{=}X_s$. (r. cont)

$\inf_n\mathbb{E}X_{t_n}>-\infty$, then $\mathbb{E}(X_t|\mathcal{F}_{s+})=\mathbb{E}(X_t|\bigcap_n\mathcal{F}_{t_n})\ge X_s$.

5) $D_n=\{\frac{k}{2^n}\}$ - submg. $X_k^{(n)}:=X_{\frac{k}{2^n}}$.

• $\tau^{(n)}=\inf\{t\in D_n: t\ge\tau\}$ is stop (!Pf)

$\downarrow\lim_n\tau^{(n)}=\tau$, r. cont. $\lim_n X_{\tau^{(n)}}=X_\tau$ a.s.

(discretized stop image in D_n)

r. cont.

$\forall K$; $\exists N$, $\forall n\ge N$, $U_\infty^{(n)}[a,b]\ge K$.

$\overline{\lim}$ $X^{(n)}$ enough near. b a $\underline{\lim}$

MCT. $\mathbb{E}U_N^{(n)}[a,b]\le\frac{\sup_t\mathbb{E}X_t^++|a|}{b-a}<\infty$.

$\mathbb{E}U_\infty^X[a,b]$

if $\sup_t\mathbb{E}X_t^+<\infty$, $\lim_{t\to\infty}X_t$ a.s. $\exists$.

$\{\underline{\lim}X_t<a<b<\overline{\lim}X_t\}\subseteq\{U_\infty^X[a,b]=\infty\}$.

1) μ on $(E^T, \mathcal{E}^T) \overset{\text{Kolmogorov.}}{\Longleftrightarrow} (\mu_I)_{I \subset \text{fin} \mathbb{R}}$ compatible.

2) Lévy process : · $X_{t_n} - X_{t_{n-1}}$ id. (ind. incre.) ($\Leftrightarrow \forall s<t$, $X_t - X_s$ $\underline{\text{id.}}$ $\mathcal{F}_s = \sigma(X_u, u \le s)$)

· (Stn.) $X_t - X_s \overset{d}{=} X_{t-s} - X_0$ · sto. cont. $t' \to t$, $X_{t'} \xrightarrow{P} X_t$.

$\Leftrightarrow$ Analytical. — convolution semigrp. $\nu_t * \nu_s = \nu_{t+s}$ $\subseteq$ C-K eq. (trasition func. p scheme brings kolmogrov. and also Markov.)

($\overset{?}{\Rightarrow}$ & $\lim_{t \downarrow 0} \nu_t = \delta_0$.) $\quad \int_E p(s,x,dy)\,p(t,y,A) = p(t+s,x,A)$

○ Brownian Motion. $\mathbb{R}^d$. 1) ind. incre.

(std. $B_0 \overset{a.s.}{=} 0$) 2) $B_t - B_s \overset{d}{=} N(0, t-s) \cdot I_d$ $\Rightarrow$ Lévy. of

$\exists$: kol, (from 1)+2) can construct cont. mod satisfying 3)) 3) a.s. orbit cont. ($P(\Omega_c) = 1$)

$\nu_t(dx) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx$ $= p_t(x)dx$ $(p(t,0,dx))$ $= \nu_t(dx - 0)$. $\{p(t,x)dx\}$.

(then $\mu_I(dx_1 \cdots dx_n) := \int_E \mu(dx)\,p(t_1, x, dx_1) \cdots p(t_n - t_{n-1}, x_{n-1}, dx_n)$ is compatible.)

($P_t * f(x)$ for heat eq. $(\frac{1}{2}\Delta - \partial_t)u = 0$.)

<Folland 10.28>

Gauss : i) a.s. orbit cont. B_t is BM

$\Leftrightarrow$ Gauss. $m(t) = 0$, $V(t,s) = t \wedge s$. ($(\frac{1}{2}\Delta - \frac{\partial}{\partial t})u = 0$ $u_0 = \delta_0$.)

ii) $B = (B^1, \cdots, B^d)$ is BM $\Leftrightarrow$ $B^i, 1 \le i \le d$ is ind. 1-dim. BM.

iii) $B_0 = 0$. a.s. cont., B_t is BM

$\Leftrightarrow \mathbb{E}\left(e^{i\langle x, B_t - B_s\rangle} \mid \mathcal{F}_s\right) = e^{-\frac{|x|^2}{2(t-s)}}$, $\forall t > s$, $x \in \mathbb{R}^d$.

iv) $\mathbb{E}|B_t|^p \approx C_p \cdot t^{\frac{p}{2}}$.

Applica...:

i) $M_t = \lambda B_{t/\lambda^2}$ is BM.

ii) $\lambda = -1$, $M_t = -B_t$ is BM.

iii) ortho. U, $M_t = UB_t$ is BM.

iv) $M_0 = 0$, $M_t = tB_{\frac{1}{t}}$ $(t>0)$ is BM.

($\lim_{t \to \infty} \frac{B_t}{t} = 0$ a.s.) for M cont. on 0.

Mg: B_t is mg.

i) $\mathbb{E}(B_t | \mathcal{F}_s) = B_s$. ii) $B_t^2 - t$ is mg.

(note $\mathbb{E}(B_t^2 - B_s^2 | \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 | \mathcal{F}_s) = t - s$)

(prog. measl. ✓ by r. cont. Strengthened flow $\sigma(\mathcal{F}_t \cup \mathcal{N})$ $\mathcal{F}$ r. cont. by Markov, then normal condition ✓)

(B_t^2 submg. Doob-Meyer decomp. $= \underline{B_t^2 - t} + t$)

iii) $e^{\gamma B_t - \frac{1}{2}\gamma^2 t}$ is mg. $\forall \gamma \in \mathbb{C}$.

($\gamma = iy$ ✓ as charac. func., then extension by dominance ($\odot$))

$M_t \xrightarrow{a.s.} 0 \Leftarrow \eta_n \xrightarrow{a.s.} 0$

$\Uparrow$ BC. Lem.

(or by BM's construction, has cont. revision)

Doob's. for η_n. $\forall \varepsilon, \sum_n P(|\eta_n| > \varepsilon) < \infty$

$:= \max_{0 \le t \le t_n} |M_t|$ $\quad P(\eta_n > \varepsilon) \le \frac{1}{\varepsilon^2} \mathbb{E}M_{t_n}^2 = \frac{t_n}{\varepsilon^2}$.

BV: $\sup_D V_D^{(p)} f = \sup_D \sum_i |f(x_i) - f(x_{i-1})|^p < \infty$

① $\sup_D V_D^{(1)} f = \lim_{|D| \to 0} V_D^{(1)} f$. ($p > 1$ ✗) ② $\lim_{|D| \to 0} V_D^{(p)} f = 0$, $\forall p > 1$. (for BV. and note ① only ✓ for (1).)

Not BV i.e. $\exists D_n$ s.t. $|D_n| \to 0$, $V_{D_n}^{(2)} f \not\to 0$.

Pf this is by the fact that $V_D^{(2)} \xrightarrow{L^2} T$, $|D| \to 0$; $V_D^{(2)}(\omega) = \sum_i \left(B_{t_i}(\omega) - B_{t_{i-1}}(\omega)\right)^2$. (a.s. ω) $[0,T]$.

(then a $V_{D_n}^{(2)} \xrightarrow{a.s.} T$.)

directly see $\mathbb{E}\left(\left[\sum_i \left((B_i - B_{i-1})^2 - \mathbb{E}(B_i - B_{i-1})^2\right)\right]^2\right) = \mathbb{E}\sum_i\left((B_i - B_{i-1})^2 - \mathbb{E}(B_i - B_{i-1})^2\right)$

$\quad [V_D^{(2)} - T]^2$ $\qquad = 2\sum_i (t_i - t_{i-1})^2 \le 2|D|T \to 0$.

Fractional BM : a Gauss process with $V(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

($\mathbb{E}|X_t - X_s|^2 = |t-s|^{2H}$) (cont. revision)

(○ Lévy : sto cont. $\Leftrightarrow \lim_{t \to 0^+} \nu_t = \delta_0$

Pf: wlog. $X_0 \triangleq 0$, ($\Leftarrow$) $\forall \varepsilon$, $\lim_{t \to 0^+} P(|X_t| > \varepsilon) = \lim_{t \to 0^+} \int 1_{\{|x| > \varepsilon\}} d\nu_t = 0$.)

($\Rightarrow$) $X_{s-t} - X_0 \sim X_s - X_t$. $\xrightarrow{P}$ implies $\xrightarrow{d}$ i.e. δ_0.

(○ Lévy conv. thm. : $\{X_n\}_{n \ge 1}$, $\varphi_n(t) \xrightarrow{p.w.} \varphi(t)$, $\varphi(t)$ cont. at 0 ($\forall t$) $\Leftrightarrow$ $\exists X_\infty$, $X_n \xrightarrow{d} X_\infty$ ✓. φ is a charac func (of X_∞)

Prokhorov Thm. $\Updownarrow$ ($\Uparrow$ ✓, $\Downarrow$ need Polish $\ni \mathbb{R}$)

$\{X_n\}_1^\infty$ tight, i.e. $\lim_{N \to \infty} \sup_n P(|X_n| > N) = 0$.)

§4

assume $\begin{cases} (\Omega, \mathcal{F}_t, \mathbb{P}) \quad (B_t) \text{ BM} \\ \mathcal{F}_t = \sigma(\mathcal{F}_t^0 \cup \mathcal{N}^0) \end{cases}$. $F_t(\omega)$ may not cont. $[0,T]$ partition D. $(t_i \to \infty)$

$I_t^D(\omega) = \sum_i f_i(\omega) \left(B_{t_{i+1}\wedge t}(\omega) - B_{t_i \wedge t}(\omega) \right)$

Lem. $f_i \in b\mathcal{F}_{t_i}$ then I_t^D is mg. (cont.)
(bounded makes intgl.)
(Note the significance of left Riem. sum)

(Doob-Meyer) $(I_t^D)^2 - A_t^D$ can be a mg. $(f_i = F_{t_i})$

where A_t^D is a cont. increasing process.

$I_t^D - I_s^D \begin{cases} ① \ f_k(B_t - B_s) \\ ② \ f_k(B_{t_{k+1}} - B_s) + \sum_i f_i(B_{t_{i+1}} - B_{t_i}) + f_j(B_t - B_{t_j}). \end{cases}$

(use tower to $s<t$, $\mathbb{E}(f_i(B_{t_{i+1}} - B_{t_i}) | \mathcal{F}_s) = \ldots \mathcal{F}_{t_i} | \mathcal{F}_s \ldots = 0$)

$\mathbb{E}((I_t^D)^2 - (I_s^D)^2 | \mathcal{F}_s) = \mathbb{E}((I_t^D - I_s^D)^2 | \mathcal{F}_s)$

$= \begin{cases} f_k^2 (t-s) \quad (B_t^2 - t \text{ is mg.}) \\ \mathbb{E}(f_k^2(t-s) + \sum_i f_i^2 (t_{i+1} - t_i) + f_j^2(t - t_j) | \mathcal{F}_s) \end{cases}$

$:= \mathbb{E}(A_t^D - A_s^D | \mathcal{F}_s)$, $A_t^D \triangleq \sum_i f_i^2 (t_{i+1}\wedge t - t_i \wedge t)$.

1. I_t^D is cont. sq. intgl. mg. $(\mathcal{M}_0^2)$
2. $I_t^{D2} - \int_0^t F_s^{D2} ds$ is mg. $\left(\sup_{0\le t\le T} \mathbb{E} M_t^2 < \infty \right)$ i.e. $\mathbb{E} M_T^2 < \infty$

simple process. $F_t^D(\omega) = \sum_i f_i(\omega) 1_{(t_i, t_{i+1}]}(t) + f_0(\omega) 1_{\{0\}}(t)$.
(no effection)
$\mathcal{L}_0$ then $A_t^D = \int_0^t (F_s^D(\omega))^2 ds$.
($f_i \in b\mathcal{F}_{t_i}$ for ada.)

$F^D \in \mathcal{L}_0 \longrightarrow I(F^D)_t \in \mathcal{M}_0^2$
$\cap$
$L^2([0,T]\times\Omega, \mathcal{B}[0,T]\times\mathcal{F}_T, dx \times \mathbb{P})$
(for F^D is var-sepl.)

(by mg. repre thm. $\overline{\{ I(F^D)_t : F^D \in \mathcal{L}_0 \}}^{\|\cdot\|_{\mathcal{M}_0^2}} = \mathcal{M}_0^2$)

now we'll see $\mathcal{M}_0^2$ is Hil sp. with $\|M\|_{\mathcal{M}_0^2} \triangleq \sqrt{\mathbb{E} M_T^2}$. (inner prod. obvious ✓)

$\mathbb{E}\, I(F^D)_T^2 = \mathbb{E} \int_0^T F_s^D(\omega)^2 ds < \infty$
$\leftarrow \quad = \|F^D\|_2$ isometry. (of 2 Hil sp.)

(Pf its completeness: Cauchy seq. $M^{(n)} \in \mathcal{M}_0^2$,
$M_T^{(n)} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$,

expand I to $\mathcal{L}^2 = \overline{\mathcal{L}_0}^{L^2([0,T]\times\Omega, \ldots)} \subseteq L^2$. (intged. process sp.)

(Fact: cont. flow. (gened by cont. process, say) gens. cont. mg. esp. Brownian)

$\exists \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ s.t. $M_T^{(n)} \xrightarrow{L^2} \xi. (M_T)$

$(I(F) \overset{\|\cdot\|_2}{=} \lim_n I(F^{(n)}).)$

• $\forall F \in \mathcal{L}^2$,

let $M_t = \mathbb{E}(\xi | \mathcal{F}_t)$.

• $G_t(\omega)$. $\sum_i G_{t_i}(\omega)(B_{t_{i+1}\wedge t}(\omega) - B_{t_i \wedge t}(\omega))$. $(F.B)_t - \int_0^t F_s^2 ds$ is mg.

(right cont. mod. is always ✓.)

(r. cont. by subseq. normaliz) $G \in \mathcal{L}^2 - ?L^2$ $\downarrow |D| \to 0$

① l. cont. adapted. $\mathcal{L}^2$-intgl.

(lessen for cond. $\mathbb{E}$) ($\underline{P} + \mathbb{E} \to \mathbb{E} \Rightarrow \underline{L}^1$)

$\mathbb{E} M_t^2 \le \mathbb{E} \xi^2$, $\forall t$. (sq. intgl.)

$\int_0^t G_s(\omega) dB_s$.

then Pf M_t is cont.: $\mathbb{E} \sup_{0\le t\le T} |M_t^{(n)} - M_t|^2 \le 4\mathbb{E} |M_T^{(n)} - M_T|$

$\sup_{0\le t\le T} |M_t^{(n)} - M_t| \xrightarrow[\text{a.s.}]{\text{subseq}} 0$; $\forall \omega \in \Omega$ a.s., $M_t^{(n)}(\omega) \xrightarrow[\to 0]{\text{uni.}} M_t(\omega)$, thus cont.

• thoughts:

$B_t^2 - T_t^D(B) \in \mathcal{M}_0^2$
$(\forall t) \ \downarrow \|\cdot\|_{\mathcal{M}_0^2} \in$
$B_t^2 - t$
(See §5)

easy facts: sub σ-alg. $\mathcal{A} \subset \mathcal{E}$, $\mu_{\mathcal{A}} = \mu|_{\mathcal{A}}$.
$L^2(E, \mathcal{A}, \mu|_{\mathcal{A}}) \subset L^2(E, \mathcal{E}, \mu)$
(closed subsp.)

$L^2(E, \overline{\mathcal{E}}, \overline{\mu})$
$\|$
$L^2(E, \mathcal{E}, \mu)$.
(if σ-fin.)

$X : [0,T]\times\Omega \longrightarrow$

① $\mathcal{B}[0,T]\times\mathcal{F}_T \ni X \not\Rightarrow X_t \in \mathcal{F}_t, \forall t$.

② $\mathcal{V} = \sigma\{\text{all } \mathcal{B}[0,T]\times\mathcal{F}_t \text{ measl.} + \text{adapted } X\}$

③ $\mathcal{M} = \sigma\{\text{prog. measl. } X\}$

④ $\mathcal{P} = \sigma\{\text{l. cont. adapted } X\}$

$\mathcal{P} \subseteq \mathcal{M} \subseteq \mathcal{V} \overset{\triangle}{\subseteq} \mathcal{P}^* \subset \overline{\mathcal{B}[0,T]\times\mathcal{F}_T}$.
(Meyer)

their $L^2 = \mathcal{L}^2$.

(for ④ l. cont. isn't close for p.w. conv. thus $\mathcal{P}$-measl. not be l. cont. ada. X.
②③ ✓)

(by ②, if $f \in \mathcal{B}$, $f(B_t) \in \mathcal{L}^2$)

$\mathbb{E}\int_0^T (F_s^D)^2 ds = \mathbb{E} S_T^2$

$F^D \xrightarrow{I} S_t \in \mathcal{M}_0^2$.
$= \sum_i F_{t_i} 1_{(t_i, t_{i+1}]}(t)$

$(= \{ (M_t)_{0\le t\le T} : \sup_{0\le t\le T} \mathbb{E} M_t^2 < \infty \})$

$(\mathbb{E} S_T^2 = \mathbb{E} A_T^D)$

(if want $T \to \infty$, and if $\sup_t \mathbb{E} M_t^2$ not $<\infty$, localize by metric ρ (cf. g_2) on $L^2([0,\infty)\times\Omega)$. thus it's not essential for $[0,T]$ or $[0,\infty)$.

§5

local martingale $\mathcal{M}_{loc}$. $M_t \in \mathcal{F}_t$, t cont., and $\exists \tau_n \nearrow \infty$ s.t. $M^{\tau_n} \in \mathcal{M}$.

(τ_n ✓, σ_n ✓ then $\sigma_n \wedge \tau_n$ ✓) $\lim_n M_t^{\tau_n} = M_t$ a.s.

($M_t^{\tau_n} \in L^1$, while may $M_t = \xi \notin L^1(\Omega)$ so better $M^{\tau_n} 1_{\{\tau_n > 0\}}$) (⟺ or $\tilde{M}^{\tau_n} \in \mathcal{M}$, i.e. $M_0 \in L^1$ omitted. $\tilde{M}^{\tau_n} = M^{\tau_n} - M_0$.)

(localize seq. of stops, $t \mapsto M_t$ mod to note for cont. (or at least càdlàg) stops $\tau_n \nearrow \infty$.) thus loc. bd.

· $M \in \mathcal{M}_{loc}$, $M^\tau \in \mathcal{M}_{loc}$. · $\mathcal{M}_{loc}$ is lin. sp.

· when $M \in \mathcal{M}_{loc}$, M is mg. ⟺ $\forall t$, $\{X_{t\wedge\tau}, \forall \tau\}$ uni. intgl. (DL-class)

(Pf: ⇒ by $M_{t\wedge\tau} = \mathbb{E}(M_t | \mathcal{F}_{t\wedge\tau})$, $\forall \tau$)

(for discrete time, $|X_{t\wedge\tau}| \le \sum_{i=0}^{t} |X_i| \in L^1$. i.e. intgl. loc. mg. is mg. for dis.)

$\mathcal{M}$ $\quad \mathcal{M}^2$ (cont.) (Hilbert) $\quad \mathcal{M}_0^2$ $\quad \mathcal{M}_b$ $\quad \mathcal{M}_{loc}$ (cont.)

• $O_M^t(|D|) = \sup_{|u-s|<|D|} |M_u - M_s|$ $(u,s \in [0,T])$. it's measl. bd. $|D| \to 0$, $O_M^t(|D|) \to 0$ a.s. ($\mathbb{Q} \to$) (if $M \in \mathcal{M}_b$)

X cont. $\quad D$: $t_0\ t_1 \ldots t_j\ t_{j+1} \ldots$, t (wlog.)

$$\forall t,\ T_t^D(X) = \sum_{i=0}^{\infty} (X_{t_{i+1}\wedge t} - X_{t_i\wedge t})^2 = \sum_{i=0}^{j} (X_{t_{i+1}} - X_{t_i})^2 + \cdots$$

$\xrightarrow[|D|\to 0]{P} A_t(\omega)$

quadratic variation process

Thm. $M \in \mathcal{M}_{loc}$. i) $\exists \langle M \rangle$

ii) $\langle M \rangle$ is the unique cont. mono↗ process s.t. $M^2 - \langle M \rangle \in \mathcal{M}_{loc}$.

iii) esp. $M \in \mathcal{M}$, ⇒ $M^2 - \langle M \rangle \in \mathcal{M}$.

Pf:

① $M \in \mathcal{M}_b$, pf $M_t^2 - T_t^D(M) \in \mathcal{M}_0^2$.

cont. ✓ mg. ✓ sq. intgl.: $\mathbb{E}(T_t^D(M))^2 < \infty$? $= \mathbb{E}\left(\sum_{i=0}^{j} (M_{i+1} - M_i)^2\right)^2 =$

② $|D| \to 0$, $M^2 - T^D(M) \in \mathcal{M}_0^2$ Cauchy seq. (2 for O_M^t control, 2 for fin. V_t control) $\le 6K^2 \mathbb{E} T_t^D(M)$

then by Jessen, $\sup_D \mathbb{E} T_t^D(M)^2 \le 36K^4$.

i.e. $\forall t$, $\| T_t^{D_n}(M) - T_t^{D_m}(M) \|_{L^2(\Omega)} \to 0$. $N_t^{n,m} \triangleq T_t^{D_n}(M) - T_t^{D_m}(M) \in \mathcal{M}_0 \cap \mathcal{M}_b$.

$\mathbb{E} T_t^D(N^{n,m}) = \mathbb{E} \sum_{i=0}^{j} (N_{r_{i+1}}^{n,m} - N_{r_i}^{n,m})^2$ $\quad D = D_n \cup D_m$. $\quad$ by ① $(N_t^{n,m})^2 - T_t^D(N^{n,m}) \in \mathcal{M}_0^2$.

$\le 2\mathbb{E}\sum_i ($ $\quad \leftarrow \quad \mathbb{E} N_t^{n,m\,2} = \mathbb{E} T_t^D(N^{n,m})$ ✓.

(uniqueness: delta since $\Delta \in \mathcal{M}_{loc}$. ×.)

now pf $\lim_{n\to\infty\ (|D|\to 0)} \mathbb{E} \sum_i (T_{r_{i+1}}^{D_n}(M) - T_{r_i}^{D_n}(M))^2 = 0$

thus $\exists \mathcal{M}_0^2 \ni N \xleftarrow[|D|\to 0]{\|\cdot\|_{\mathcal{M}^2}} M^2 - T^D(M)$.

$=$ $\quad \le \mathbb{E}(4 O_M^t(|D|)^2 T_t^D(M))$

$\langle M \rangle_t \triangleq M_t^2 - N_t$, $M_t^2 - T_t^D(M) \xrightarrow{L^2} M_t^2 - \langle M \rangle_t$.

③ $M \in \mathcal{M}_{loc}$, $\tau_n \nearrow \infty$, $M^{\tau_n} \in \mathcal{M}_b$. by Hölder. ✓

$\exists \langle M^{\tau_n} \rangle_t$ is M^τ's qv. and $(M^{\tau_n})^2 - \langle M^{\tau_n} \rangle \in \mathcal{M}_0^2$ unique. cont.↗.

$\langle M^{\tau_{n+1}} \rangle_t = \langle M^{\tau_n} \rangle_t$ (↑a.s.)

$T_t^D(M^{\tau_{n+1}}) = T_t^D(M^{\tau_n})$, $t \le \tau_n$. then can def $\langle M \rangle_t^{(\omega)} \triangleq \langle M^{\tau_n} \rangle_t^{(\omega)}$, $\forall \tau_n > t$, $\forall t$.

$=$ $\quad$ (is cont. mono↗) ✓ $(M^2 - \langle M \rangle)^{\tau_n} = (M_t^{\tau_n})^2 - \langle M^{\tau_n} \rangle_t \in \mathcal{M}$

then pf $\forall t$, $T_t^D(M) \xrightarrow{P} \langle M \rangle_t$. $\quad \in \mathcal{M}_{loc}$.

④ $M \in \mathcal{M}^2$ ⇒ $M - \langle M \rangle \in \mathcal{M}$.

by DL class ← control $M_{t\wedge\tau}^2 - \langle M \rangle_{t\wedge\tau}$.

$\sup_\tau M_{t\wedge\tau}^2 \le \sup_{0\le s\le t} M_s^2 \in L^1$ (Doob)

$\sup_\tau \langle M \rangle_{t\wedge\tau} \le \langle M \rangle_t \in L^1$.

§6

Step:

1. $L^2(M) \subseteq L^2(M^\sigma)$ Thm.

$\|\cdot\|_{L^2(M^\sigma)} \le \|\cdot\|_{L^2(M)}$

2. $\langle M^\sigma\rangle = \langle M\rangle^\sigma$

3. $(F.M)^\sigma = F.M^\sigma$.

(by $(F.M)^\sigma \overset{\mathcal{M}_0^2}{=} \lim_n (F^{(n)}.M)^\sigma$

$\mathcal{L}_0 \ni F^{(n)} \to F$ $= \lim_n \sum_i F^{(n)}_{t_i}(M^\sigma_{t_{i+1}} - M^\sigma_{t_i})$

$\overset{\mathcal{M}_0^2}{=} F.M^\sigma$.)

covariation process $\langle M,N\rangle := \frac{1}{4}(\langle M+N\rangle - \langle M-N\rangle)$

$\langle M,N\rangle \overset{P}{=} \frac{1}{4}\lim_n \sum_i$

$M\in\mathcal{M}_0^2$, $F,G \in \mathcal{L}^2(M)$.

σ

$(F.M)^\sigma = F.M^\sigma$

$\langle M,N\rangle^\sigma = \langle M^\sigma,N^\sigma\rangle = \langle M, N^\sigma\rangle = \langle M^\sigma, N\rangle$

$F.M \longleftrightarrow \langle M,N\rangle$ $\quad \forall M,N\in\mathcal{M}_{loc}$.

$\langle F.M, G.N\rangle = FG.\langle M,N\rangle$

$\iff (F.M)(G.N) - (FG).\langle M,N\rangle \in \mathcal{M}_0$

$\iff \forall$ bd. σ, $E(F.M)^\sigma_T (G.N)^\sigma_T = E\int_0^T F_tG_t\,d\langle M,N\rangle^\sigma_t$

i.e. $E(F.M^\sigma)_T(G.N^\sigma)_T \overset{?}{=} E\int_0^T F_tG_t\,d\langle M^\sigma,N^\sigma\rangle_t$.

$(F\in\mathcal{L}^2(M)\to F\in\mathcal{L}^2(M^\sigma))$

$\mathcal{L}_0 \ni F^{(n)} \xrightarrow{\mathcal{L}^2(M)} F$

$E(F^{(n)}.M)_T(G^{(n)}.N)_T \overset{\checkmark}{=} E\int_0^T F^{(n)}_t G^{(n)}_t\, d\langle M,N\rangle_t$

$(n\to\infty)$ $\quad \downarrow \mathcal{M}_0^2$ $F.M$, $\quad \downarrow \mathcal{M}_0^2$ $G.N$, $\quad \downarrow L^1$ $\quad \downarrow$ K-W Ineq.

$E(F.M)_T(G.N)_T \qquad E\int_0^T F_tG_t\, d\langle M,N\rangle_t$.

$\langle M^\sigma,N^\sigma\rangle = \langle M^\sigma, N\rangle$

$\iff M^\sigma N^\sigma - \langle M^\sigma,N\rangle \in \mathcal{M}_{loc}$

$\iff M^\sigma N^\sigma - M^\sigma N \in \mathcal{M}_{loc}$

$T_n \nearrow \infty$, $T_n = \inf\{t : |M_t|\ge n \text{ or } |N_t|\ge n\}$.

$\iff (M^\sigma N^\sigma - M^\sigma N)^{T_n}\in\mathcal{M}_0$

$\iff M^{T_n\wedge\sigma}N^{T_n\wedge\sigma} - M^{T_n\wedge\sigma}N^{T_n} \in \mathcal{M}_0$

$[\tilde M \triangleq M^{T_n},\ \tilde N \triangleq N^{T_n} \in \mathcal{M}_b]$

$\iff \tilde M^\sigma\tilde N^\sigma - \tilde M^\sigma \tilde N \in\mathcal{M}_0$

$\iff \forall$ bd. τ, $E(\tilde M_{\tau\wedge\sigma}\tilde N_{\tau\wedge\sigma}) = E(\tilde M_{\tau\wedge\sigma}\tilde N_\tau)$

✓. $E(\tilde M_{\tau\wedge\sigma}\tilde N_{\tau\wedge\sigma}) = E(E(\tilde M_{\tau\wedge\sigma}\tilde N_\tau|\mathcal{F}_{\tau\wedge\sigma}))$

esp. $G\equiv 1$, then $\langle F.M, N\rangle = F.\langle M,N\rangle$

and the last one part:

$F^\sigma.M^\sigma \overset{?}{=} (F.M)^\sigma$.

K-W's construction of sto. intg.

(repre of fund.)

$F\in L^2([0,T]\times\Omega, \mathcal{M}, \mu)$, $M\in\mathcal{M}_0^2$.

$|\phi_F(N)| = |E\int_0^T F_t(\omega)\,d\langle M,N\rangle_t|$

$\le E$

Riesz. then $\exists!\ K\in\mathcal{M}_0^2$, $\phi_F(N) = EK_TN_T$

$= E(F.N)_TN_T$.

§7

- $M \in \mathcal{M}_{loc}$, $\mathcal{L}^2_{loc}(M) \triangleq \{ F \text{ prog.} : \exists \tau_n \nearrow \infty, \text{ s.t. } \mathbb{E}\int_0^{\tau_n} F_t^2 \, d\langle M\rangle_t < \infty, \forall n \}$.

For loc.

$= L^2_{loc}([0,\infty)\times\Omega, \mathcal{M}, \mu_{\langle M\rangle}) \supseteq \mathcal{B}_{loc} := \{F \text{ prog, loc. bd.}\} \supseteq \mathcal{B}_{loc.\ell} := \{F \text{ l.cont. adapted. loc.bd.}\}$
$\supseteq \{F \text{ cont. ada.}\}$.

$(\Rightarrow F \in \mathcal{L}^2(M^{\tau_n}) \Rightarrow F.M^{\tau_n} \in \mathcal{M}_0^2)$

to def its intg.: $(F.M)_t \triangleq (F.M^{\tau_n})_t$, $\forall t \le \tau_n, \forall n$. (well-def by $n<m$.

then fill the graph of eq.s.

$$\begin{array}{ccc} & \tau & \\ \langle M,N\rangle^\tau = \langle M,N^\tau\rangle = \langle M^\tau,N\rangle = \langle M^\tau,N^\tau\rangle & & (F.M)^\tau = F.M^\tau = F^\tau.M^\tau \\ \langle M,N\rangle & \longleftrightarrow & F.M \end{array}$$

① $\langle F.M, G.N\rangle = FG.\langle M,N\rangle$.

② $F.M$ is the unique $\in \mathcal{M}_{loc}$ s.t. $\langle F.M, N\rangle = F.\langle M,N\rangle, \forall N \in \mathcal{M}_{loc}$.

Pf:
left $= \langle (F.M)^{\tau_n}, (G.N)^{\tau_n}\rangle$
$= \langle F.M^{\tau_n}, G.N^{\tau_n}\rangle \overset{\checkmark}{=} FG.\langle M^{\tau_n}, N^{\tau_n}\rangle$
$\in \mathcal{M}_0^2$ $= FG.\langle M,N\rangle^{\tau_n}$.

- Cont. semimg.: $X_t = M_t + V_t$. V_t is BV. (loc.) (ada.) $(V_0 \equiv 0)$ a.s. · lin.sp. · unique decomp. $(M \perp BV)$

$(F.X)_t \triangleq (F.M)_t + (F.V)_t$.

$\cdot \langle X\rangle_t \triangleq \underset{|D|\to 0}{\overset{P.}{\lim}} \sum_{i=0}^{n} [(M_{t_{i+1}\wedge t} - M_{t_i \wedge t})^2 + 2 ⓜⓥ + ⓥ^2]$ $\overset{P.}{=} \langle M\rangle_t$. $\xrightarrow[a.s.]{} 0$ (control by $O_\omega(|D|)$)

F wrt. X intgl. $\Leftrightarrow F \in \mathcal{L}^2_{loc}(M) + \exists \tau_n \nearrow \infty$, s.t. a.s. ω, $F_t^{\tau_n}(\omega)$ wrt. $V_t^{\tau_n}(\omega)$ intgl. · $\langle X, Y\rangle = \langle M^X, M^Y\rangle$.

$\Leftarrow F \in \mathcal{B}_{loc.} \Leftarrow F \in \mathcal{B}_{loc.l.} \supseteq \{\text{cont. ada.}\}$

$F \in \mathcal{B}_{loc.l.}$, X cont. semimg. $\Rightarrow \underset{|D|\to 0}{\lim} \sum_i F_{t_i}(X_{t_{i+1}} - X_{t_i}) \overset{P.}{=} (F.X)_t$.
$(D = \{0 = t_0 < \dots < t_n = t\})$

$F_t^D(\omega) = \sum_{i=0}^{n-1} F_{t_i}(\omega) 1_{(t_i, t_{i+1}]}(t) + 1_{\{0\}}(t) F_0(\omega) \xrightarrow[\forall t,\omega]{} F_t(\omega)$.

Thm. X csmg. $F^{(n)}, F \in \mathcal{B}_{loc.}$ s.t. ① $F^{(n)} \xrightarrow[(t,\omega)]{\forall t,\omega} F$ ② $\exists G \in \mathcal{B}_{loc.}$, s.t. $|F^{(n)}| \le G$ $\forall t,\omega,n$

then $(F^{(n)}.X)_t \xrightarrow{P.} (F.X)_t, \forall t$.

Pf: ① G bd. $M \in \mathcal{M}_0^2$.

(actually ①'s $\forall t,\omega$ can be relaxed to $\mu_{\langle M\rangle}$-a.e. but not $\forall t$, μ-a.e. ω only if $\langle\langle dt \times P\rangle$)

- Ito's Formula. $F \in C^2(\mathbb{R}^d)$, $X = (X^d)$ csmg.

then $F(X)$ is csmg. and $F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\, d\langle X^i, X^j\rangle_s$.

① $F = x^2$ ② $F = x^n$ ③ F = multinomial. (multivar) ④ X bd. $\exists K$ s.t. $|X_t(\omega)| \le K$. $\exists F^{(n)}$, $\begin{cases} F^{(n)} \xrightarrow{C[-K,K]} F \\ F^{(n)\prime} \rightrightarrows F' \\ F^{(n)\prime\prime} \rightrightarrows F'' \end{cases}$ (or S-W Thm.) ⑤ $(\Uparrow)$

⑤ $\tau_n := \inf\{s > 0 : |X_t| > n\} \nearrow \infty$. X^{τ_n} csmg.
$|X^{\tau_n}| \le n$.
$F(X_t^{\tau_n}) - F(X_0^{\tau_n}) = \int_0^t F'(X_s^{\tau_n})\, dX_s^{\tau_n} + \frac{1}{2}\int_0^{t\wedge\tau_n} F''(X_s)\, d\langle X\rangle_s$

$F'(X^{\tau_n}).X^{\tau_n} = F'(X)^{\tau_n}.X^{\tau_n} = (F'(X).X)^{\tau_n}_t$.

esp.
($\Delta f = 0$, $f(B_t)$ is $\mathcal{M}_{loc}$)

- further depiction of UI. $(X_t)_{t\ge 0}$. $(X \in \mathcal{M}_0^2)$

$\Rightarrow X_t \xrightarrow[L^1]{a.s.} X_\infty$; $X_t = \mathbb{E}(X_\infty | \mathcal{F}_t)$ $[0,\infty]$.

$\Rightarrow$ ① $\forall \tau$ stop. $X_\tau = \mathbb{E}(X_\infty | \mathcal{F}_\tau)$,

② $\forall \sigma \le \tau$, $X_\sigma = \mathbb{E}(X_\tau | \mathcal{F}_\sigma)$.

(note there needn't bd. stops)

(① ⇒ ② obvious.)

Pf of ①:

§8

- description of BM

Lévy

Gauss $\mathcal{M}_0^2$

$\mathbb{E}X_sX_t = s\wedge t$.   (Lévy desc.)

Thm. $(M_t)_{t\ge0}$ Brown. $\Leftrightarrow$ $M_t^i \in$ cont. $\mathcal{M}_{loc}$. $+ \langle M^i, M^j\rangle_t = \delta_{ij}t$, $\forall i,j$. ($M_t^iM_t^j - \delta_{ij}t$ is mg.)

Pf.

§10

O-U equation $dX_t = -cX_t dt + \sigma dB_t$.   ($\mathbb{R}^\infty$ (Hil). 标准 Brown $\mathbb{E}\left|\sum_{i=1}^\infty W_t^{(i)} e_i\right|^2 = +\infty$ 发散) 但 O-U 有一 $e^{-()}$ 压缩因子.

$\underline{d=1}$ $dX_t = \sigma(t,X_t)dB_t + b(t,X_t)dt$.   (strictly to say,

$X_t \in \mathbb{R}$ (Δ) $X_t - X_0 = \int_0^t \sigma(t,X_t)dB_t + \int_0^t b(t,X_t)dt$)

$\sigma, b : [0,+\infty)\times\mathbb{R} \to \mathbb{R} : \mathcal{B}[0,+\infty)\times\mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathbb{R})$

($\sigma(X_t)$: Markov 型, $\sigma(t)$: Itô 型/时齐)

2 cond. i) X_t is ada. cont.

ii) $\int_0^t |b| dt < +\infty$, a.s. $\forall t$. (可交换)

Set-up: ① $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, B_t)$. B_t is $\mathcal{F}_t$ - std. BM. and! $B_t - B_s$ ind. with $\mathcal{F}_s$. $\forall t > s \ge 0$.

② X_t satisfies i), ii) and (Δ), a.s. $\forall t$.

$F_s \triangleq \sigma(s, X_s)$, $F_s \in \mathcal{L}^2_{loc}(B) = \{\exists \tau_n \nearrow \infty$, s.t. $\mathbb{E}\int_0^{\tau_n} F_s^2 ds < +\infty\}$.

$\Leftrightarrow \int_0^t F_s^2 ds < +\infty$, a.s. $\forall t$.   ($\Rightarrow$: $\forall \omega\in\Omega$ let $\tau_n \nearrow +\infty$ a.s. $\Leftarrow$: $\tau_n = \inf\{t\ge0 : \int_0^t F_s^2 ds > n\}$.)

(note $t \mapsto \int_0^t F_s^2 ds$ ada. cont.) (by simple approx.)

thus ii) $-+\int_0^t F_s^2 ds < +\infty$.

(X, B) is a weak solution.   (then $\sigma(B)$ ind. $\mathcal{F}_0$)

Uniqueness:
- Dis. unique: $\forall (X,B), (X',B')$ s.t. $X_0 \overset{d.}{=} X_0' \Rightarrow X \overset{d.}{=} X'$ (fin. dim. dis.) ($\supseteq \sigma(X_0)$)
- Orbit unique: Same B_t. (structure) $X_0 \overset{a.s.}{=} X_0' \Rightarrow X_t \overset{a.s.}{=} X_t'$, $\forall t$.

$\Uparrow$ $\underline{\text{Y-W}}$.

$\beta_\omega(t) \triangleq B_t(\omega)$. $\beta_\omega \in C([0,+\infty),\mathbb{R}) \triangleq \mathcal{W}^1$. Polish and its top. $\mathcal{B}(\mathcal{W}^1)$.   proj. $\pi_t(\omega) = \omega(t)$,

$\rho(\omega_1,\omega_2) = \sum_{n=1}^\infty \frac{1}{2^n}$ (sup $\mathcal{B}_t(\mathcal{W}^1) = \sigma(\pi_s : s \le t)$.

(X, B) Strong solution: $\exists F : \mathbb{R}\times\mathcal{W}_0^1 \to \mathcal{W}^1$. s.t. $\mathcal{W}_0^1 \triangleq C_0([0,+\infty),\mathbb{R})$ (starts at 0)

unique sg. solution: $\exists F$

① F satisfies $\underline{\text{some}}$ measl. ($\overset{?}{\sim} X_0$ 96' Kallenberg)

② $X(\omega) = F(X_0(\omega), B(\omega))$. a.s. ω.

(Y-W 引')

Sg. 2nd.: F is a sg. solution s.t.   unique by $\forall (X,B)$ satisfies $X = F(X_0, B)$

i) $\forall D \in \mathcal{B}_t(\mathcal{W}^1)$. $F^{-1}(D) \in \mathcal{B}(\mathbb{R})\times\overline{\mathcal{B}_t(\mathcal{W}_0^1)}$.   ($\Rightarrow G(x,B) = F(x,B)$.)

ii) $\forall$ struct. B. $\xi \in \mathcal{F}_0$, $X = F(\xi, B)$ is the solution satisfying $X_0 = \xi$.

需验证 X_t 满足 i. ii.

① $0 < T < +\infty$. $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\le T}, \mathbb{P}, W_t)$. $\xi \overset{d.}{=} \mu$ ind. W_t.
dis. μ