泛实变小记.

Week1 映射的复合、粘接、乘积. 集族的选择函数. Axiom of Chioce

偏序关系、极大元、上界. Zorn's Lemma. (e.g. at least inj. $X\to Y$ or $Y\to X$)
($\to$ card. 三歧性)

Week2 集合对等(等势) $\exists$ bij. $f: X\to Y$, $|X|=|Y|$. e.g. $\chi: 2^X \to \{0,1\}^X$, $A \mapsto \chi_A$.

$X_i \simeq Y_i$, $\coprod_i X_i \simeq \coprod_i Y_i$; $\prod_i X_i \simeq \prod_i X_{\sigma(i)}$ (重排).

无限集等价命:

e.g. 交换群同构 $\mathbb{Z}[t] \simeq \mathbb{N}$.

bij. $\sum_k a_k t^k \longmapsto \prod_k p_k^{a_k}$ (prime (p_k)) $\in \mathbb{Q}_+$.

A

i) 含A的集合均无限.

ii) A含一个可列集B. (thus $\aleph_0$ is smallest card. inf)

iii) $A \simeq A\cup C$, 当C可数时. ($A\sqcup C$ also)

iv) (Dedekin inf) $A \simeq A$的真子集A_1. (by bij. $f: A\sqcup C \to A$, $f(A)\subset A$)
($f^n(C)$ countl.)

e.g. 代数数 $\mathbb{A} \simeq \mathbb{N}$.

Cantor-Bernstein Thm. 反称性.

$\forall (f,n)$ surj. to $\mathbb{A}$. $f\in \mathbb{Z}[x]$. (n-th root of f)

Pf: 取 inj. $f: X\to Y$, $g: Y\to X$, $h=gf: X\to X$.

找 A, B s.t. $\begin{cases} X = A\sqcup g(B) \\ Y = f(A)\sqcup B\end{cases}$, $X\simeq Y$. let $C = X\setminus g(Y)$, $A=\bigcup_{n\geq 0} h^n C$, $B = Y\setminus f(A)$

then $X\setminus g(B) = X\setminus (g(Y)\setminus gf(A))$
$= (X\setminus g(Y)) \cup h(A) = A$.

(Folland: $x \to g^{-1}(x) \to f^{-1}(g^{-1}(x))\dots$ terminates at X or Y or inf.) (可根据此反猜A形式,
sep into X_∞, X_X, X_Y
$f\downarrow Y_\infty, \downarrow Y_X, Y_Y \uparrow g$. 由于 $h(A)\subset A$)

非退化区间 $\simeq$ 连续统. $T=[-1,1]\subset \mathbb{R}\subset \overline{\mathbb{R}}$,

$f: x\longmapsto \dfrac{x}{1-|x|}$, $T\simeq \overline{\mathbb{R}} \simeq \mathbb{R}$.

$|X| < 2^{|X|}$, $\aleph = 2^{\aleph_0}$

且 $[a,b] \simeq T$.

(inj. $\mathbb{R}\to 2^{\mathbb{Q}}$; inj. $2^{\mathbb{N}}\to\mathbb{R}$ by $A\mapsto \sum_{n\geq 0}\frac{\chi_A(n)}{3^n}$)

($\mathbb{R} = \mathbb{A}\sqcup\mathbb{T} = \mathbb{Q}\sqcup\mathbb{J}$. thus $\mathbb{T}$ and $\mathbb{J}$ are continuum)

基数运算 $2\leq a\leq\aleph$ 时, $a^{\aleph_0}=\aleph$; $1\leq a,b\leq\aleph_0$ 时, $a+\aleph = a\aleph = \aleph^b = \aleph$.

e.g. $\dim_{\mathbb{Q}}\mathbb{R} = \aleph$

(hamel基的基数)

维数定理:

i) $\forall$ Hamel基 C, D, $|C|=|D| \triangleq \dim_{\mathbb{K}} X$.

ii) $|\mathbb{K}| < \dim X$ 时, $|X| = \dim X$.

(Pf: i) 用C表示D, $|C|\leq \sum_\alpha |C_\alpha| \leq |D|\aleph_0$ (设D infinite)

ii) X 划分成 $\{\sum_{e\in F} c_e e \mid c_e\in\mathbb{K}, |F|=n, F\subset D\}$, $|X| = \sum_n |\mathbb{K}|^n \dim X^n$
吸收得 $\dim X$)

Week3

序同构、序稠密、(上/下)有界集、(上/下)定向集、完备序(格)、遗传集、单调完备序、共尾子集。

(A预序集中的B s.t. $\forall a\in A, \exists b\in B, a\le b$)

网、子网. $\overline{\lim} = \inf_{i\in A}\sup_{j\in A, j\ge i} x_j$. $\chi_{\overline{\lim}_{i\in I} T_i} = \overline{\lim}_{i\in I}\chi_{T_i}$.

($\inf_{j\in B} x_j \le \underline{\lim}_{j\in B} x_j \le \overline{\lim}_{j\in B} x_j \le \sup_{j\in B} x_j$, 单调时 $\lim$ 为 $\sup_{j\in B}$ or $\inf_{j\in B}$.)

$\overline{\mathbb{R}}^+$下Σ的结合律、累次求和、乘法公式。

($A=\coprod_i A_i$) MCT、DCT 均成立。(函数网极限 p.w.)

(note $c_i\ge 0$时, Cauchy sum $\lim_{n\to\infty}\sum_{i=0}^{n} c_i = \sum_{n\ge 1} c_n \triangleq \lim_{A\nearrow_{fin} B}\sum_{i\in A} c_i = \sup_{A\nearrow_{fin} B}\sum_{i\in A} c_i$ (B)

since $\sup_n \sum_{i=0}^{n} c_i \le \sup_{A\in fin B}\sum_{i\in A} c_i$.) ($\le$ with $\max A$)

Tarski不动点:

完备序集$(L,\le)$有最大&小元a, b(完备格), $f: L\to L$ 单增, $\exists x\in L$ s.t. $f(x)=x$.

Observation: 区间G, H. $|G|_n\le |H|_n$ if $G\subseteq H$.

i) $|H|_n = \sup\{|G|_n : \text{closed } G\subseteq H\}$ (内正则)

ii) $\forall s>0$, $\exists$可数有界开区间A_i覆盖H, $\sum_i |A_i|_n \le |H|_n + s$.

iii) $\coprod_{\alpha\in A} H_\alpha \subset \bigcup_{\beta\in B} H_\beta$ 且B可数, 则 $\sum_\alpha |H_\alpha|_n \le \sum_\beta |H_\beta|_n$.

($\coprod_\alpha H_\alpha = \coprod_\beta H_\beta$, A, B均可数时相等) (Pf:

Week4

(集)环、半环、代数。 $\mathcal{P}$简单分解其生成的集环$\mathcal{R}(\mathcal{P})$ 即$\mathcal{P}$为半环。

($\setminus^{\&}\coprod_{fin.}$ or $\cup$ <=> $\cap^{\&}_{fin.}\Delta$ or $\cup_{fin.}$)

$E\subseteq F$, $\mu(E)\le\mu(F)$. $E=\bigcup_i E_i$, $\mu(E)\le\sum_i\mu(E_i)$ ($i\in I$ fin.)

容度/测度: i) $\mathcal{R}(\mathcal{P})$上唯一扩张 ii) 单调 iii) 有限次加.

(暂指半环$\mathcal{P}$上)

测度: 可列加的容度. i) $E_n\nearrow E$, $\lim\mu(E_n)=\mu(E)$. ii) 下连续 iii) 可列次加. iv) 上连续 $E_n\searrow E$, $\exists\mu(E_k)<\infty$, $\lim\mu(E_n)=\mu(E)$. (取$E\setminus E_n$.)

(<=>无穷集下连续+空集上连续)

单调环<=>σ-环. (单调类+集环)   Borel. F_σ, G_δ. $\mathcal{B}=\mathcal{S}$(区间)

tip: $\mathcal{B}(X)\otimes\mathcal{B}(Y)\subseteq\mathcal{B}(X\times Y)$ 且X, Y第二可数时$=$. (取拓扑基)

好集原理: $f: X\to Y$, $\mathcal{F}\subseteq 2^Y$, $f^{-1}(\mathcal{F}) := \{f^{-1}(B) \mid B\in\mathcal{F}\}$, then $\mathcal{S}(f^{-1}(\mathcal{F})) = f^{-1}(\mathcal{S}(\mathcal{F}))$. (拉回)

Pf: $f^{-1}(\mathcal{S}(\mathcal{F}))$是一个$\sigma$-环; $f(\mathcal{S}(f^{-1}(\mathcal{F})))$是一个$\sigma$环.

$\subseteq$ $\supseteq$

(basic idea: pf that $\mathcal{A}$ has property P. $=\sigma(\mathcal{E})$)

$\begin{cases}\mathcal{E}\subset\mathcal{B}\\ \mathcal{B} \text{ is } \sigma\text{-alg.}/P \text{ preserve under oprations.}\end{cases}$ e.g. 单调类方法.

$\mathcal{B}=\{A \mid A\in\mathcal{A} \text{ with } P(A)\}$)

(将生成系对应即可)

Week 5

单调类定理：集环 $\mathcal{A}$，$\mathcal{M}_0(\mathcal{A})=\mathcal{S}(\mathcal{A})$. λ-类定理 (Dykin π-λ).

problem: want to expand μ on $\mathcal{P}$ ($\mathcal{R}(\mathcal{P})$) to $\mathcal{S}(\mathcal{P})$?

(遗传6-环) $\mathcal{H}(\mathcal{P})$ 上 $\mu^*: E\mapsto \inf\{\sum_{i=1}^{\infty}\mu(A_i)\mid A_i\in\mathcal{P}, E\subseteq\bigcup_{i=1}^{\infty}A_i\}$. 有:

i) $\mathcal{R}(\mathcal{P})\subseteq\mathcal{S}(\mathcal{P})\subseteq\mathcal{H}(\mathcal{P})$. ii) μ^* 为外测度 iii) $E\in\mathcal{R}(\mathcal{P})$, $\mu^*(E)=\mu(E)$

($\mathcal{H}(\mathcal{P})=\{E\mid\exists A_i\in\mathcal{P}, E\subseteq\bigcup_{i=1}^{\infty}A_i\}$) (非负、单调、可列次加) iv) $\mu^*(E)=\min\{\mu^*(F)\mid E\subseteq F\in\mathcal{S}(\mathcal{P})\}$.

(E∈$\mathcal{H}(\mathcal{P})$) (note on $\mathcal{S}$ it's meas.)

problem': μ^* is not meas. and need finite additive to become. restrict to $\mathcal{S}(\mathcal{P})\subseteq\mathcal{P}^*\subseteq\mathcal{H}(\mathcal{P})$.

△ Carathéodory: (可割测量 <=> 有限可加)

$\mu^*(F)=\mu^*(F\cap E)+\mu^*(F\setminus E)$, $F\in\mathcal{H}(\mathcal{P})$ 只要对 $F\in\mathcal{P}$ ($\mu(F)<\infty$) 成立.

μ^*-可测集 $\mathcal{P}^*$ ($\supseteq\mathcal{P}$) 为完全测度的6-环. ($\supseteq\mathcal{S}(\mathcal{P})$)

Pf: i. 证 $F\in\mathcal{P}$ 即可. $\left.\begin{array}{l}\mu^*(F\cap E)\le\mu^*(F_1\cap E)+\cdots\\ \mu^*(F\setminus E)\le\mu^*(F_1\setminus E)+\cdots\end{array}\right\}$ $\mu^*(F\cap E)+\mu^*(F\setminus E)\le\mu(F_1)+\cdots$ 取右侧下确界即 $\mu^*(F)$.

ii. $\mathcal{P}^*$ 为6-环. (含测度证明) $\mu^*(F\cap(E_1\setminus E_2))+\mu^*(F\cap(E_1\setminus E_2)^c)$ 反向由可列次加性.

$\underset{E_1\text{割}F\cap(E_1\setminus E_2)^c}{=}\mu^*(F\cap E_2^c\cap E_1)+\mu^*(F\cap E_1\cap E_2)+\mu^*(F\cap E_1^c)$

(iii. μ^*-null set $\subseteq\mathcal{P}^*$) $\underset{E_1,E_2\text{割}}{=}\mu^*(F)$. (并)

$E=\coprod_i E_i$, $\mu^*(F)=\sum_{i=1}^{n}\mu^*(F\cap E_i)+\mu^*(F\cap(\coprod_{i=1}^{n}E_i)^c)\ge\sum_{i=1}^{n}\mu^*(F\cap E_i)+\mu^*(F\cap E^c)$

令 $F=E$ 即得可列加性. $\underset{n}{\lim}\ \ge\mu^*(F\cap E)+\mu^*(F\cap E^c)$ 故取等.

或由有限加性得) (可列次加性)

(注意利用上面iii), iv)将 μ^* 放至 $\mathcal{P}$, $\mathcal{S}(\mathcal{P})$ 上, 利用测度连续性和加性)

property: i) $E_n\in\mathcal{H}(\mathcal{P})$, $\lim_{n\to\infty}\mu^*(E_n)=\mu^*(\lim_{n\to\infty}E_n)$

ii) $E_n\in\mathcal{H}(\mathcal{P})$, $\mu^*(\varliminf_{n\to\infty}E_n)\le\varliminf_{n\to\infty}\mu^*(E_n)$; $E_n\in\mathcal{P}^*$, $\exists k$ s.t. $\mu^*(\bigcup_{i\ge k}E_i)<\infty$, $\varlimsup_{n\to\infty}\mu^*(E_n)\le\mu^*(\varlimsup_{n\to\infty}E_n)$

iii) $\lim_{i\to I}\mu^*(E_i\Delta E)=0$, $E_i\in\mathcal{P}^*$ then $E\in\mathcal{P}^*$.

iv) (简单逼近) $E\in\mathcal{P}^*$, $\mu^*(E)<\infty$, $\forall r>0$, $\exists A\in\mathcal{R}(\mathcal{P})$ s.t. $\mu^*(E\Delta A)<r$.

($\sum_{i=1}^{\infty}\to\sum_{i=1}^{n}\mu(A_i)<\mu^*(E)+r$)

Cara. Uniqueness: $A_i\in\mathcal{P}$ and $\bigcup_{i=1}^{\infty}A_i\in\mathcal{S}(\mathcal{P})$ measl.

测度 μ 在 $\mathcal{P}$ 上，其在 $\mathcal{S}(\mathcal{P})$ 上有扩张 μ'，当 μ 6-有限时唯一.

(Pf: $\mathcal{M}_0=\{E\in\mathcal{S}\mid\nu(E\cap H)=\mu'(E\cap H)\}$, $H\in\mathcal{P}$ 且 $\mu(H)$ 有限. 为单调环.)

(不妨可先扩至集环 $\mathcal{P}$ 上) (μ) 故 $\nu(E)=\mu'(E)$ by 可列加性, $\{H_i\}$ 为覆盖.

6-有限的刻画: 0) μ^* 的6-有限可令 $A_i\in\mathcal{P}$ 覆盖. (in $\mathcal{H}(\mathcal{P})$) $\mu(A_i)<\infty$

i) μ^* 的6-有限 E, $\exists F\in\mathcal{S}(\mathcal{P})$ s.t. $E\subseteq F$, $\mu^*(F\setminus E)=0$ $\in\mathcal{P}^*$

ii) $\mathcal{S}\setminus\mathcal{N}=\mathcal{S}\cup\mathcal{N}\triangleq\mathcal{P}'$ is a 6-环;

iii) μ 6-有限 <=> μ^* 6-有限 (on $\mathcal{H}(\mathcal{P})$), 且此时 $\mathcal{P}'=\mathcal{P}^*$.

Week 6. ($=\mathcal{J}_n^*$)

$(\mathcal{B}_n\subset)\ \mathcal{L}_n$ Lebesgue 测度. Cantor 集. (无处稠密. 闭. $m(G)=0$. $|G|=\aleph$.)

Cantor 函数 $g(x)$ (连续↑)(eg. $h: x\longmapsto x+g(x)$ $[0,1]\to[0,2]$ 同胚 note $m(h(G))=1$.)

(与拓扑的)正则性: $E\subseteq\mathbb{R}^n$,

(Leb 外测度的外正则) $m^*(E)=\inf\{m(B)\mid E\subseteq \text{open}\, B\subseteq\mathbb{R}^n\}$.

等价: i) (Leb 测度的内正则) $m(E)=\sup\{m(A)\mid \text{comp.}\,A\subseteq E\}$.

E可测, ii) $\varepsilon>0$, $\exists$ open $B\supseteq E$ s.t. $m(B\setminus E)<\varepsilon$. iv) $\varepsilon>0$, $\exists$ closed $D\subseteq E$ s.t. $m(E\setminus D)<\varepsilon$.

iii) $\exists$ G_δ-set $B_0\supseteq E$ s.t. $m(B_0\setminus E)=0$. v) $\exists$ F_σ-set $D_0\subseteq E$ s.t. $m(E\setminus D_0)=0$.

(pf: $|B_n|<|A_n|+\frac{\varepsilon}{2^n}$, bd. $A_n\in\mathcal{J}_n$) (iii), v) 对 $\mathbb{R}^n\setminus E$) (σ-comp.)

(ii), iv) 可修为 m^* 以推出 E可测; (note use σ-fin of m to $\coprod_k E_k$ then for each E_k)

iii), v) 自然可测了. ($\Rightarrow$ i) by iv) sup $|D|$, let $D_n=D\cap\{x\le n\}$, $D_n\uparrow D$ for comp.)

(E可测且测度有限, 等价: vi) $\varepsilon>0$, $\exists$ comp. $A\subseteq E$ s.t. $m(E\setminus A)<\varepsilon$.

and m^* for pf this. vii) $\varepsilon>0$, $\exists$ $F\in\mathcal{R}(\mathcal{J}_n)$ s.t. $m(E\Delta F)<\varepsilon$.)

f 连续, Borel 集 $B\subseteq Y$, $f^{-1}(B)$ 为 X 的 Borel 集; (好集原理. 单保 $\cap$、$\setminus$)

f 单射且开/闭, Borel 集 $A\subseteq X$. $f(A)$ 为 Y 的 Borel 集.

Vitali 集 $x\sim y$ if $x-y\in\mathbb{Q}$. $m(A)\le\sum_{u\in B} m(u+A_\tau)\le m(B+A)<\infty$,

chosen $\tau:\tilde{A}\to A$, $A_\tau:=\tau(\tilde{A})$. ($B=\mathbb{Q}\cap(A-A)$) leads to $m(A)=0$ ※!.

i) $A_\tau\notin\mathcal{L}$ if A 有界 $\in\mathcal{L}$ 且 $m(A)>0$.

ii) $A\in\mathcal{N}$ iff. $\forall T\subset A$, $T\in\mathcal{L}$. (∃非Borel的Leb可测集. $2^{\aleph}$ ($\mathbb{R}^n$ 上 $\mathcal{B}$ 共 $\aleph$ 个)

平移: $m^*(x+F)=m^*(F)$ e.g. $h(G)_\tau\notin\mathcal{L}$, $h^{-1}(h(G)_\tau)\in\mathcal{N}$)

(双射 $f(\mathcal{P})=\mathcal{Q}$ 且 $A\in\mathcal{P}$ 有 $\mu(A)=\nu(f(A))$, (连续 f 将 $\mathcal{L}_m$ 映入 $\mathcal{L}_n$ iff. $\mathcal{N}_m$ 映入 $\mathcal{N}_n$,

则 $f(\mathcal{P}^*)=\mathcal{Q}^*$ 且 $A\in\mathcal{H}(\mathcal{P})$ 有 $\mu^*(A)=\nu^*(f(A))$.) since σ-紧已成立)

(thus $\lim_{x\to0} m((E+x)\Delta E)=0$ by $F\in\mathcal{R}(\mathcal{J}_n)$ with $m(E\Delta F)\to0$)

缩放. 线性算子: $m^*(T(F))=|\det T|\,m^*(F)$ on $\mathbb{R}^n$.

乘积: $\mathcal{L}_k\otimes\mathcal{L}_l\subset\mathcal{L}_{k+l}$. first for $A\in\mathcal{B}(\mathbb{R}^k)$, $B\in\mathcal{B}(\mathbb{R}^l)$, $A\times B\in\mathcal{B}(\mathbb{R}^{k+l})$,

$A\longmapsto |A\times(a,b]|_{k+l}$ & $|A|_k|(a,b]|_l$ by Cara. same on $\mathcal{B}(\mathbb{R}^k)$.

$A_i:=A\cap\{|x|\le i\}$. $B\longmapsto|A_i\times B|_{k+l}$ & $|A_i|_k|B|_l$ by Cara. same.

(suppose $T\notin\mathcal{L}_l$, thus $\subsetneq$) $A_i\uparrow A$, $A_i\times B\uparrow A\times B$ to get $|A\times B|_{k+l}=|A|_k|B|_l$.

for null, $|N\times T|\le|A\times\mathbb{R}^l|=0$ ($N\subset A\in\mathcal{B}(\mathbb{R}^k)$)

then $\forall\mathcal{L}$ we have $\underbrace{N\cup A}_{=E}$, $\underbrace{M\cup B}_{=F}$ to get the same product eq.

(then $E\times F\in\mathcal{L}_{k+l}$, $|E\times F|=|E||F|$)

(X_i, Y_i 需为 X, Y 的可测子空间)

Week7

可测映射的复合、粘接、De积：

i) 截口保持可测集、可测映射. $V \in \mathcal{R}_1 \otimes \mathcal{R}_2$, $V_a \in \mathcal{R}_2$, $V^b \in \mathcal{R}_1$.

meas'l. $f: V \to W$, $f_a: V_a \to W$, $f^b: V^b \to W$ meas'l.

ii) $Y = \prod_i Y_i$, $\mathcal{T} = \bigoplus_i \mathcal{T}_i$, 则 $\mathcal{T}$ 有生成系 $\{\pi_i^{-1}(B_i) \mid B_i \in \mathcal{T}_i\}$.

($\oplus$ 定义成 $\prod_i E_i$ 且可数个 $E_i \neq X_i$)

iii) $f: (X, \mathcal{R}) \to (Y, \mathcal{T})$ 可测 iff $\pi_i f$, $\forall i$ 可测;

$f_i: (X_i, \mathcal{R}_i) \to (Y_i, \mathcal{T}_i)$ 可测 则 $\prod_i f_i: \prod_i (X_i, \mathcal{R}_i) \to (Y, \mathcal{T})$ 可测.

至广义实轴 $\bar{\mathbb{R}}$ 上的带号函数.

(拓扑子基 $[-\infty, a)$, $(b, +\infty]$)

(生成系 $\{(-\infty, b), \{+\infty\} \mid b \in \bar{\mathbb{R}}\}$)

关于可测函数以下等价:

i) f 可测. 此时 f^+, f^-, $|f|$ 均可测.

ii) $\forall b \in \bar{\mathbb{R}}$, $\{f < b\}$ 可测. (by 好集原理, 且 $\{f = \pm\infty\} = \left(\bigcup_{b \in \mathbb{Q}} \{f < b\}\right)^c$ meas'l.)

(or $\{f > a\}$ etc.)

(f_i 均可测) ($\sum f_i$ also)

iii) f 为 $\sup_i f_i$, $\inf_i f_i$, $\overline{\lim_i} f_i$, $\underline{\lim_i} f_i$.

iv) f 为 $f_i \pm f_j$, f_i / f_j, $f_i f_j$.

(iii) $\to$ i): $\{f \le a\} = \bigcup_i \{f_i \le a\}$ for $\sup_i f_i$.

$\overline{\lim_i} = \inf_i \sup_{j \ge i} f_j$. also.

iv) $\to$ i): $\{f < b\} = \bigcup_{r < \mathbb{Q}} (\{f_i < a\} \cap \{f_j < b - r\})$,

Borel $\phi_{(t)} = -t$ or $\frac{1}{t}$. $f_i^+ f_j^+$ also.)

e.g. 递增函数为 Borel 函数且间断点第一类、可数.

for 复值 $\mathbb{C}^n$, f 可测时 $\mathrm{re} f$, $\mathrm{im} f$, $\mathrm{sgn} f$, $|f|$ 可测,

其余性质类似. ($\mathrm{sgn}(0) = 0$)

Week8

(一致) 连续的集合表达. e.g. $\bigcap_n \bigcup_j \bigcap_{i \ge j} \{|f_i - f| < 2^{-n}\}$, $\bigcap_n \bigcap_{i \ge j_n} \{|f_i - f| < 2^{-n}\}$.

简单逼近: meas'l. f on $(X, \mathcal{M})$.

$\exists$ p.w. $\to f$ 的简单函数列, 且使 $|f_n| \xrightarrow{p.w.} |f|$.

当 $\sup |f|(X) < \infty$ 时, 一致收敛 ($\sum_{n \ge 1}$ 一致可积).

($f \ge 0$ 时可写 $f = \sum_{n \ge 1} c_n \chi_{X_n}$) (cf by $X_{nk} = \{k-1 \le 2^n (f \wedge 2^n) \le k\}$

$f = \lim_{n \to \infty} \sum_{k \ge 1} c_{nk} \chi_{X_{nk}}$.)

Luzin Thm.: $X \in \mathcal{L}$, f on X meas'l.

$\delta > 0$, $\exists$ closed $E \subset X$ s.t. $m(X \setminus E) < \delta$, $f|_E$ cont. / f_n is simple, $0 \le f - f_n \le \frac{1}{2^n}$, $\forall x$.

Pf: f simple at $X_i \setminus A_i$, $E = \coprod_i A_i$, $f|_E$ ✓. (不是 f 在 E 上 cont.!)

($\forall y \in O(x, r) \cap E$, 其中 $O(x, r) \subset (\coprod_{j \ne i} A_j)^c$.)

then $\frac{\delta}{2^k}$ for E_k, $f_k \to f$ (uni. conv. for cont. since 可以用个连续函数把 $\bar{\mathbb{R}}$ 压进有界 $\frac{x}{1+|x|}$ e.g.)

① a.u. $\to$ ② n.u. $\to$ ③ m.

② $\to$ ④ a.e. (a.p.w.)

let $E_0 = \bigcap_n E_{\frac{1}{n}}$.

③: $\forall \varepsilon, \forall \delta, \exists k^{(\varepsilon)}, i \ge k$, $\mu(|f_i - f| \ge \varepsilon) < \delta$.

$\overline{\lim_i} \mu(|f_i - f| \ge \varepsilon) \le \delta$ ✓ $\to 0$

③ $\longleftrightarrow$ $\exists$ 子列 ②: ($\Leftarrow$) $c_n = \mu(|f_i - f| \ge \varepsilon)$. $c = \overline{\lim_n} c_n$.

($\forall$ 子列的~)

$\exists$ subseq. $c_{n_i} \to c$, $\exists c_{n_j}$ subsubseq. with $f_j \xrightarrow{n.u.} f$, $c_{n_j} \to 0$.

(Riesz: $\xrightarrow{m.}$ 有子列 $\xrightarrow{a.e.}$)

($\Rightarrow$) (可取该子列为列)

$i \ge k_n$, $\mu(|f_i - f| \ge \frac{1}{2^n}) < \frac{1}{2^n}$. then $c = 0$.

$E_j = \bigcup_{n \ge j} \{|f_{k_n} - f| \ge \frac{1}{2^n}\}$, $\mu(E_j) \le \frac{1}{2^{j-1}}$.

then $E_{j(\delta)}^c$ 上 n.u.

④ → ② 当有限 (X,μ): $F_n = \bigcap_k \underbrace{\bigcup_{i\ge k} \{|f_i - f| \ge 2^{-n}\}}$

(Erogov Thm.)

thus $\exists k_n$, $\mu(E_n^{k_n}) < \frac{\delta}{2^n}$. $E_n^k \downarrow F_n$, $\mu(F_n)=0$.

(以上收敛 a.e.唯一)

$E = \bigcup_n E_n^{k_n}$, $\mu(E)<\delta$.

$X\setminus E = \bigcap_n \bigcap_{i\ge k_n} \{|f_i - f| < 2^{-n}\}$. ✓ (p.w. on E^c)

Littlewood 三原理.

Week 9

$L(f|E) = \sup_D L(f,D)$, $U(f|E) = \inf_D U(f,D)$. D 为可测分解.

有 i) D' finer of D, $L(f,D) \le L(f,D')$, $U(f,D') \le U(f,D)$.

(取 $D \vee D'$ 得) $L(f,D) \le U(f,D')$. esp. $L(f|E) \le U(f|E)$.

ii) $L(f,D) = \sum_i L(f,D_i)$ if $D = \sqcup_i D_i$; $L(f|E) = \sum_i L(f|E_i)$ if E_i 可测分解 E.

f 可测, 有 $\forall \varepsilon>0$, $\exists D$ s.t. $U(f,D) \le L(f,D)+\varepsilon$, esp. $L(f|E)=U(f|E)$. (记 $\int_X f d\mu$)

(pf: $A_n = \{2^n < f \le 2^{n+1}\}$, $A_{ni} = A_n \cap \{i\delta_n \le f < (i+1)\delta_n\}$.

for 1-4. ($f: X\to \mathbb{R}^+$) $\sup f(A_{ni})\cdot\mu(A_{ni}) \le \inf f(A_{ni})\cdot \mu(A_{ni}) + \delta_n \cdot \mu(A_{ni})$. Σ).

性质: 1. $\forall a \overset{+\infty}{\ge} 0$. $\int_X af d\mu = a\int_X f d\mu$. 2. $\int_X (f_1+f_2)d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu$.

(2'). $g \le h$, $\int_X g d\mu \le \int_X h d\mu$. 3. $\int_X \chi_E d\mu = \mu(E)$ (取 $D = \{E, X\setminus E\}$) (取 $D = D_1 \vee D_2$. $\sup f(A) \le \sup f_1(A) + \sup f_2(A)$.)

推广 $f \in L^1 \iff |f| \in L^1$. $f: X \to \mathbb{C}$, $\int_X f d\mu := \int_X \mathrm{re} f d\mu + i\int_X \mathrm{im} f d\mu$.

(1.2. 在有意义时仍✓) 4. $\int_X f d\mu < +\infty$, $\mu(f=+\infty)=0$ 且 $\{f>0\}$ σ-fin.; $\int_X f d\mu = 0$, $\mu(f>0)=0$. (对各 A_n 讨论即得)

5. $\lim_{n\to\infty} \int_{E_n} f d\mu = \int_E f d\mu$ 上/下连续 ($\int_{E_1} < +\infty$) (meas. $\nu: E \mapsto \int_E f d\mu$) by ii). 6. (3'.) $\int_\emptyset f d\mu = 0$. thus 修改 null 集上值 不改积分.

Cheb. ineq. $\varepsilon \mu(|f| \ge \varepsilon)^{\frac{1}{p}} \le \|f\|_p$. 7. $\left|\int_X f d\mu\right| \le \int_X |f| d\mu$.

Week 10.

8. 内正则性. $f \in L^1$,

$\forall \varepsilon>0$, $\exists$ fin μ E s.t. $\sup |f|(E) < +\infty$ (bd.) 且 $\int_{X\setminus E} |f| d\mu < \varepsilon$.

Pf: 8. $\{|f|>0\}$ σ-fin. (去掉 $\{|f|=\infty\}$, 其可测∞)

i.e. $E_n \nearrow X$ ($\setminus \{|f|=\infty\}$ null)

$F_n := E_n \cap \{|f| \le n\}$. $\nearrow X$.

下连续 $\nu(F_n) \nearrow \nu(X)$.

9. 绝对连续性. ..,

.., $\exists \delta>0$ s.t. $\forall \mu(F)<\delta$ 时 $\int_F |f| d\mu < \varepsilon$.

Pf: 9. $\nu(F) = \nu(F\cap E) + \nu(F\cap E^c)$

$< (1+\sup f(E))\,\mu(F) + \underbrace{\nu(E^c)}_{<\varepsilon}$ (by 8.)

$$\delta := \frac{\varepsilon - \nu(E^c)}{1+\sup f(E)}. \checkmark$$

Thm. Riemann可积. (<=> Darboux Thm.) $[a,b]$ 上 f(实值)有界 <=> a.e. cont. 此时 Leb. intg. = Rie. intg.

Pf: ($\Leftarrow$) 令 $C = \sup|f|(X)$ 为界. 取 Rie. 分点组 (x_i) by

间断点集 $D = \bigcup_k D_k$. D 上外延到 $m(V) < \varepsilon$, open V. D^c 上 cont. 点…对应

$D_k = \{x \mid \forall \delta > 0, \exists y, |y - x| < \delta, |f(y) - f(x)| \geq 2^{-k}\}$. 的构成区间 T_i. $\sum_i m(T_i) < \varepsilon$. + $|f(y) - f(x)| \leq \varepsilon$ 的 V_x. closed.

$X \cap V_x \to \leq (b-a) \cdot 2\varepsilon$.

Darboux, $\leftarrow X \cap T_i$ —— $\leq 2C \cdot \varepsilon$ f_{in} 有限覆盖 排列得到 端点.

$\leq 2(C + b - a)\varepsilon$ ✓.

($\Rightarrow$) 已有分点组 (x_i). δ_j s.t. $\omega_j \geq 2^{-k}$.

$$m(D_k) \leq m(\{x_i\} \cup \delta_j) \leq \sum_j \delta_j \leq 2^k \sum_j \omega_j \delta_j < 2^k \varepsilon \to 0.$$

此时, 各 δ_i 上 $\omega_i = \sup_i f - \inf_i f$ $\quad \inf_i f \cdot \delta_i \leq \begin{matrix}\text{Leb}_i \\ \text{Rie}_i\end{matrix} \leq \sup_i f \cdot \delta_i$ i.e. $|\text{Leb}_i - \text{Rie}_i| \leq \omega_i \delta_i \to 0$

(a.e. cont. s.t. Leb. measl.) $\quad |\text{Leb} - \text{Rie}| < \varepsilon$.

内闭可积的(形式) Leb. 反常积分.

积分可列加性的等价形式 ($\lim$ 与 $\int$ 的交换命题).

1) Monotone. f_n measl. $\nearrow f$. 且 f_k 积分不为 $-\infty$ (以除以 f_1^- 讨论 ≥ 0 时)

$\lim_{n\to\infty} \int_X f_n d\mu = \int f d\mu$. Pf: $\leq$ ✓. $\geq$: 先除去 f_1^- in A,

$\int_X f_n d\mu \geq \int_{A \cap \{f_n \geq rf\}} f_n d\mu + \int_{B \cap \{f_n \geq M\}} f_n d\mu$ $\quad A = \{f < +\infty\}$, $B = \{f = +\infty\}$. $r \in (0,1)$, 有 $A \cap \{f_n \geq rf\} \nearrow A$, $B \cap \{f_n \geq M\} \nearrow B$. $f_n \geq rf$, $n \to +\infty$. pw.

cont. $\lim_n \int_X f_n d\mu \geq r \int_A f d\mu + M \int_B d\mu$, $r \to 1^-$. ✓. (1)' $f_n \searrow f$, f_k 积分不为 $+\infty$. also.)

2) Sum. $\int_X \sum_{n\geq1} |f_n| d\mu = \sum_{n\geq1} \int_X |f_n| d\mu$ (by 1)); if it's fin., $\sum_{n\geq1} f_n \xrightarrow{a.e.} f$ 且 $\int_X f d\mu = \sum_{n\geq1} \int_X f_n d\mu$

- measl. f can be $= \sum_{n\geq1} c_n \chi_{E_n}$. 单调类方法.

Pf: $g = \sum_{n\geq1} |f_n|$ intgl. i.e. a.e. fin.

$\exists f = \sum_{n\geq1} f_n$ (a.e.), $|f| \leq g$. f intgl. 且

$\left|\int f d\mu - \sum_{i\leq n} \int f_i d\mu\right| \leq \int \left|f - \sum_{i\leq n} f_i\right| d\mu \leq \int \sum_{i>n} |f_i| d\mu = \sum_{i>n} \int |f_i| d\mu \to 0$.

3) Fatou. g^- intgl., $f_n \overset{a.e.}{\geq} g$.

$\int_X \varliminf_{n\to\infty} f_n d\mu \leq \varliminf_{n\to\infty} \int_X f_n d\mu$.

Pf: $h_n := \inf_{m\geq n} f_m$, $h := \varliminf_n f_n$

(g^+ intgl. $f_n \overset{a.e.}{\leq} g$, $\int_X \varlimsup_n f_n d\mu \geq \varlimsup_n \int_X f_n d\mu$) $\quad h_n \nearrow h$. thus $\int h = \lim_n \int h_n = \varliminf_n \int h_n$

$\int \varliminf_n f_n = \quad \leq \varliminf_n \int f_n$. ($h_n \leq f_n$).

4) Bounded. (net f_i) $f_i \xrightarrow{\mu} f$ 且 $|f_i| \overset{a.e.}{\leq} c$.

fin. (X,μ) $\lim_i \int_X |f_i - f| d\mu = 0$ (esp. $\lim_i \int_X f_i d\mu = \int_X f d\mu$)

Pf: $\int_X |f_i - f| d\mu \leq \varepsilon \cdot \mu(|f_i - f| < \varepsilon) + \int_{|f_i - f| \geq \varepsilon} 2c\, d\mu \to 0$.

$(\leq \mu(X) < +\infty)$

5) Dominance. (Ver①) $f_n \xrightarrow{a.e.} f$, $|f_n| \overset{a.e.}{\leq} g \in L^1$ (note for 4), 5), we have $|f| \overset{a.e.}{\leq} g$.)

$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$ (eq. ...)

(Ver②) (net) $f_i \xrightarrow{\mu} f$, $|f_i| \overset{a.e.}{\leq} g \in L^1$. ...

Pf: ① $\varlimsup_n \int |f_n - f| d\mu \leq \int \varlimsup_n |f_n - f| d\mu = 0$. (or C)

② $\int_X |f_i - f| d\mu \leq \int_E |f_i - f| d\mu + \int_{X\setminus E} 2|g| d\mu$

(E 为 g 的 ...) ↑ 4) bd. $\varlimsup_i \int_X |f_i - f| d\mu \leq 2\cdot\varepsilon \to 0$.

- $f_n \geq 0$. $f_n \xrightarrow{\mu} f$. $\int f_n \to \int f$.

则 $f_n \xrightarrow{L^1} f$. $(f_n + f)^+ \in L^1$ 即可, note $(f - f_n)^+ \leq f$.)

$\xrightarrow{\mu} 0$.

6) Para. Int. $g(x) = \lim_{x\to x_0} \int f(x,y)\mu(dy) = \int f(x_0, y)\mu(dy)$, ($|f(x,y)| \leq g(y) \in L^1$) $g'(x_0) = \int \frac{\partial f}{\partial x}(x_0, y)\mu(dy)$. ($\left|\frac{\Delta_x f(x,y)}{\Delta x}\right| \leq g(y) \in L^1$.)

$\phi: (X, \mathcal{R}, \mu) \to (Y, \mathcal{S}, \nu)$.

$\nu(F) = \mu(\phi^{-1}(F))$ (ϕ hom. 保测变换)

可由 μ 诱导 ν (ϕ measl.)

(or ν 推送回生成 X 上的测度结构)

乘积测度

半环 $\mathcal{S}_1 \circ \cdots \circ \mathcal{S}_n$ 上有测度 $\mu: \prod_i E_i \mapsto \prod_i \mu_i(E_i)$

(矩形) 作 Cará 扩张得 $\otimes \mathcal{S}_i$ 上测度. $\mu := \mu_1 \times \cdots \times \mu_n$

Week 11

已知: 截口、(函数截口) 保持可测性. 欲证:
(及投影)

σ-fin. $(X, \mathcal{R}, \mu)$ 与 $(Y, \mathcal{S}, \nu)$ 乘积空间 $(X\times Y, \mathcal{R}\otimes\mathcal{S}, \mu\times\nu)$ 上可测集 A, 有

$y \mapsto \mu(A^y)$, $x \mapsto \nu(A_x)$ 可测; $\mu\times\nu(A) = \int_Y \mu(A^y)\nu(dy) = \int_X \nu(A_x)\mu(dx)$.

Pf: $Z_n := X_n \times Y_n$ 有限 $\nearrow X \times \nearrow Y$. A 为矩形等式成立, 进而 $\mathcal{M}_0 = \{A \mid A \cap Z_n \text{满足等式}\}$ (且边长有限) $\supset \mathcal{R}(\mathcal{R}\circ\mathcal{S})$ 环.

now 单调列 $A_k \to A$, $\mu((A \cap Z_n)^y) = \lim_k \mu((A_k \cap Z_n)^y)$ (cont.) 故可测,

DCT 故等式成立. then $\mathcal{M}_0 \supseteq \mathcal{R}\otimes\mathcal{S}$. then again (cont.)

$\mu(A^y) = \lim_n \mu((A\cap Z_n)^y)$ 故可测.

MCT 故等式成立.

Tonelli–Fubini Thm. $x \mapsto \int_Y f(x,y)\nu(dy)$,

$\int_{X\times Y} f d(\mu\times\nu) = \int_X d\mu \int_Y f d\nu = \int_Y d\nu \int_X f d\mu$. $y \mapsto \int_X f(x,y)\mu(dx)$ 可测.

(σ-fin. X, Y)

(Pf: $f = \sum_i c_i \chi_{E_i}$ for ≥ 0, then for Fubini. f intgl. iff. |f| 的某个重积分有限)

Week 12

$\mathcal{L}^p$-norm. i) $\|af\|_p = |a|\,\|f\|_p$. $\|f\|_p = 0$ iff. $f \overset{a.e.}{=} 0$.

$(0 < p \leq +\infty)$ ii) Hölder. $\frac{1}{p} = \sum_i^n \frac{1}{p_i}$, $f = \prod_i^n f_i$, then $\|f\|_p \leq \prod_{i=1}^n \|f_i\|_{p_i}$. ($0<p<1$ 时有 Hölder, Minkow. 反向)

iii) $(1 \leq p \leq +\infty)$ Minkowski. $\|g+h\|_p \leq \|g\|_p + \|h\|_p$.

Pf: ii) 可设 $0 < \|f_i\|_{p_i} < +\infty$, $|f_i| < +\infty$, 进一步 $\frac{f_i}{\|f_i\|_{p_i}}$ 可设 $\|f_i\|_{p_i} = 1$, $\|f\|_p \leq 1$?

$|f|^p = \prod_{i=1}^n (|f_i|^{p_i})^{\frac{p}{p_i}}$

AM-GM. $\leq \sum_{i=1}^n \frac{p}{p_i} |f_i|^{p_i}$

(log 的凹性)

$\int |f|^p \leq \sum_i \frac{p}{p_i} \cdot \underbrace{\int |f_i|^{p_i}}_{1} = 1$. (可去 $\|f_j\|_\infty$ 部分)

iii) $\int |f|^p \leq \int |f|^{p-1}|g| + \int |f|^{p-1}|h| \leq \|g\|_p \underbrace{\left(\int |f|^{(p-1)q}\right)^{\frac{1}{q}}}_{\text{设}\neq 0, <+\infty} + \|h\|_p \cdots$

($f = g + h$)

可设 $|g|, |h| < +\infty$.

$\mathcal{L}^p$ 完备性: $f_n \xrightarrow{\mathcal{L}^p} f$ iff. $\{f_n\}$ $\mathcal{L}^p$-Cauchy. 此时若 $f_n \in \mathcal{L}^p$, f 也 $\in \mathcal{L}^p$.

$\lim_n \|f_n - f\|_p = 0$ $\quad$ $\lim_{i,j} \|f_i - f_j\| = 0$ $\quad$ (Minkowski.)

Pf: (⇒) Minkowski. (⇐) f_n 依测度 Cauchy. $E_n := \{|f_{k_{n+1}} - f_{k_n}| \geq 2^{-n}\}$ ($m, l \geq k_n$ 时)

$X \setminus E$ 上 (p.w.) Cauchy $f_{k_n}(x) \to f(x)$. $\quad$ 子列 + BC. $\mu(\underbrace{\varlimsup_n E_n}_{:=E}) = 0$.

$= \bigcup_m \bigcap_{n \geq m} \{|f_{k_{n+1}} - f_{k_n}| < 2^{-n}\}$ (补充 $x \in E$ 时 $f(x) = 0$)

$(|f_{k_m} - f_{k_n}| < \sum \cdots \to 0)$ $\quad$ Fatou $\lim_n \int |f_n - f|^p \leq \varlimsup_n \varliminf_m \int |f_n - f_{k_m}|^p \to 0$.

简单逼近 $\quad$ 半环 $\mathcal{P}$ 使 $\mathcal{S} \subseteq \mathcal{P}^*_\mu$. $f \in \mathcal{L}^p$, $\forall \varepsilon > 0$, $\exists$ simple $g = \sum_{i=1}^n a_i \chi_{E_i}$ s.t. $\|g - f\|_p < \varepsilon$.

(原函数 simple 逼近 p.w. 由 DCT, $f_n \xrightarrow{\mathcal{L}^p} f$. $E_i \in \mathcal{R}(\mathcal{P})$ s.t. $\mu(E_i \Delta F_i) < \delta_i$.

$f_n = \sum_i^n b_i \chi_{F_i}$ ($\leq 2|f|^p$) $< \frac{\varepsilon}{2}$ $\quad$ $\|f_n - g\| \leq \sum_i^n b_i \|\chi_{F_i} - \chi_{E_i}\|_p = \sum_i b_i \delta_i^{\frac{1}{p}} < \frac{\varepsilon}{2}$)

连续逼近 $\quad$ $X \subseteq \mathbb{R}^n$ $\in \mathcal{L}eb.$ $\quad$ $f \in \mathcal{L}^p$, $\forall \varepsilon > 0$, $\exists$ cont. $g: X \to \mathbb{C}$ s.t. $\|g - f\|_p < \varepsilon$.

(simple $h < \frac{\varepsilon}{2}$. for E_i, 内外正则性 $B_i \supset E_i$, $A_i \subset E_i$, $\mu(B_i \setminus A_i) < \delta_i$.

(补充至 $\mathbb{R}^n$)  $g_i(x)$ 可取 $\frac{d(x, \mathbb{R}^n \setminus B_i)}{d(x, A_i) + d(x, \mathbb{R}^n \setminus B_i)}$ $\quad$ $\|g_i - \chi_{E_i}\|_p \leq \delta_i^{\frac{1}{p}}$

Week13 广义测度、(全)正/负集 $\quad$ $\|g - h\|_p \leq \sum_i^n |a_i| \delta_i^{\frac{1}{p}} < \frac{\varepsilon}{2}$.)

signed $\mu: \mathcal{S} \to \overline{\mathbb{R}}$, $\mu(\bigsqcup_i E_i) = \sum_i \mu(E_i)$. $\quad$ $\mathcal{S}^+ := \{A \in \mathcal{S} \mid \forall E \in \mathcal{S}, \mu(A \cap E) \geq 0\}$, $\mathcal{S}^-$ 为 σ-环.

(不可能同时可取 $\pm\infty$) $\quad$ Hahn 分解. $X^+ \sqcup X^-$. $\quad$ $(\setminus ✓, \bigsqcup_i ✓)$

$\mu^+(E) := \mu(X^+ \cap E)$, $\mu^-(E) := -\mu(X^- \cap E)$. $\quad$ (差0集下唯一)

(全变差测度) $|\mu| = \mu^+ + \mu^-$. $\mu = \mu^+ - \mu^-$ Jordan 分解. $\quad$ $|\mu|$ fin iff. μ^+, μ^- fin. iff. μ fin.

complex $\mathrm{re}\,\mu$, $\mathrm{im}\,\mu$ signed. $\quad$ E 为 $|\mu|$ 的 σ-fin. iff. $\cdots$ $\mu^\pm$ iff $\cdots$ μ.

$|\int f d\mu| \leq \int |f| d|\mu| (\leq \|f\|_\infty \|\mu\|)$. (or def by $|\mu|: E \mapsto \sup\{\sum_{A_i} |\mu(A_i)| : \{A_i\}$ 可测分解 $E\}$)

f wrt. μ intgl. iff. $|f|$ wrt. $|\mu|$ intgl. $\quad$ thus $|\mu|(E) \geq |\mu(E)|$.

DCT also. ($f_n \xrightarrow{|\mu|} f$, $|f_n| \leq g \in \mathcal{L}_{|\mu|}$).

(weighted) $\nu: E \mapsto \int_E f d\mu$. then $\int_X g d\nu = \int_X g f d\mu$, 记 $d\nu = f d\mu$.

(is meas.) $\quad$ (by let $g = \chi_E$ ✓)

性质: i) (链式) $d\lambda = g d\nu$, $d\nu = f d\mu$ 时 $d\lambda = g f d\mu$. $\quad$ 故 $f \neq 0$ 时 $d\mu = \frac{1}{f} d\nu$.

ii) $d|\nu| = |f| d|\mu|$. $\quad$ esp. $d\mu = h_\mu d|\mu|$.

Pf: i) 验证 $g = \chi_E$ 时. ii) $f = \sum_i c_i \chi_{A_i}$. $\quad$ ($h_\mu := \chi_{X^+_\mu} - \chi_{X^-_\mu}$, $d|\mu| = h_\mu d\mu$)

$|\nu|(X) = \sup\{\sum_i \sum_{D_j} |\nu(D_j)| : D_j$ 可测分解 $A_j\} = \sum_j |c_j| |\mu|(A_j)$.

R-N 导数. $\nu \ll \mu$. μ σ-fin. $\exists f\in\mu$ s.t. $d\nu = f\,d\mu$ 且 $|\mu|$-唯一. ν σ-fin. 时 f (a.e.) 有限.

Pf: (≤:) $H := \{g\in\mu \mid \int_E g\,d\mu \le \nu(E), \forall E\in\mathcal{S}\}$. $f=\sup H$, 有 $f\,d\mu \le d\nu$ (MCT) 且 $\int f\,d\mu = \sup_{g\in H}\int g\,d\mu$. (取一列 g_n 到 sup.)

(≥:) Hahn (P,Q) of $(f+c)d\mu - d\nu$. (因此 ν σ-fin. 时 $\mu\{f=\infty\}=0$)

$\Rightarrow \underbrace{(f+c\chi_Q)}_{\in H}\,d\mu \le d\nu$. 故 $\mu(Q)=0$, $(f+c)\,d\mu \overset{a.e.}{\ge} d\nu$, $c\to 0$.

Leb. 分解. ν,μ σ-fin. (唯一地) ν_c,ν_s s.t. $\nu=\nu_c+\nu_s$, $\nu_c\ll\mu$, $\nu_s\perp\mu$.

Week 14

全变差 $\bigvee_X f := \sup\{\sum_i^n |f(t_i)-f(t_{i-1})| \mid \{t_i\}_0^n 划分X\}$. 性质: $|\cdot|$的性质.

(半)连续性: $f_i \xrightarrow{p.w.} f$, $\bigvee f \le \varliminf_i \bigvee f_i$. $X_i\nearrow X$, $\bigvee_X f = \lim_i \bigvee_{X_i} f$.

跳跃函数 $h(b)-h(a) = \sum_{r\in D} p_r\chi_{(a,b]}(r) + \sum_{r\in D} q_r \chi_{[a,b)}(r)$
(left jump) (right jump)

($h\in$Borel 且 $h'\overset{a.e.}{=}0$.)

cont. g s.t. $V(g+h) = Vg+Vh$.

$f^V: x\mapsto \bigvee_a^x f$. BV $\ni f = g+h$ (Pf: $|f(x+)-f(x)| = f^V(x+)-f^V(x)$. 取出间断点, 作 h. 则 $g=f-h$ 连续)

Jordan分解 $g^{V+}-g^{V-}+g_0$. BV <=> 2单调函数之差. ($\mathbb{C}$则4个)

Helly: $X\subseteq\mathbb{R}$. (已知单增 $g_n: X\to\mathbb{R}$ 有 p.w. 收敛子列)

有界且 $\sup_n \bigvee f_n$ 有限 (by g_n on $\mathbb{Q}\to g$, expand to $\mathbb{R}$ with 可数集 D 间断. 在 $\mathbb{R}\setminus(\mathbb{Q}\cup D)$ 上连续)

$f_n: X\to\mathbb{C}$ 有子列 $\xrightarrow{p.w.} f\in BV$. (拆成上面已知.)

Week 15

Dini导数. (Leb.) ($X\subseteq\mathbb{R}$上) (单增.) BV a.e. 可导. (Fubini 逐项求导) f_n 单增. $\sum_n f_n$ p.w. 有 Cauchy 和 f. 则 $f' \overset{a.e.}{=} \sum_n f_n'$

绝对连续. (线性组合, p.w. 乘积 封闭)

等价有: i) f 为 (某可积函数) Leb. 不定积分 (原函数). 此时必是 f' (a.e. 意义). (Leb. 可导)

$f:[a,b]\to\mathbb{C}$. ii) f 有界变差. 此时 $\bigvee_a^b f = \int_a^b |f'|\,dx$.

(即 Leb. 基本定理 calculus) iii) f 绝对连续. 且 f^V, $|f|$ 均绝对连续.

Leb. 分解. $f_{ac.}+f_{s.}+f_d$ (⊥) (jump) ($f_s'\overset{a.e.}{=}0$)

Pf: i)→ii) from $\int|f'|\,dx = \int (f^V)'\,dx \le \bigvee f$, $\bigvee\int u \le \|u\|_1$.

由连续逼近得到 $f' \overset{a.e.}{=} u \leftarrow g' = \nu$, cont. ν.

ii)→iii) 取等知 $f^V(x) = \int^x |f'|\,dx$. 对$|f'|$ 由积分的绝对连续性即 $f_c - \int f_c'\,dx$.

iii)→i) let $f \mapsto f - \int f'$ (可积性见左), f^V ac. $\Rightarrow f$ ac.

由 i) (此时) 可设 $f'\overset{a.e.}{=}0$, 证 f const.

(note ac. has a.e. f'.)

$\int_x^y \frac{f^V(t+r)-f^V(t)}{r}\,dt = \int_y^{y+r}\frac{f^V(t)\,dt}{r} - \int_x^{x+r}\frac{f^V(t)\,dt}{r} \le f^V(y+r)-f^V(x)$.

Fatou 得. 且 $(f^V)' \overset{a.e.}{\ge} |f'|$.

$B\supset E = \{D_- f = 0\}\setminus\{b\}$, $\underbrace{(s,t)\subset B}_{\Sigma<\delta}$ s.t. $f(t)-f(s)\le \varepsilon(t-s)$.

Lebesgue diff. Thm. $\nu\in L_{loc}$. a.e. Leb点, $\lim_{r\to0}\int_x^{x+r}\left|\frac{\nu(s)-\nu(x)}{r}\right|ds = 0$.

记 $g(x) = \int^x \nu(t)\,dt$. g 在 ν 的 Leb 点, 有 $g' = \nu(x)$. (⇐不成立)